Ph.D. Qualifying Exam: Linear Algebra, Probability Distributions and Statistical Inference

Thursday June 15, 2006

Problem 1. (a) Let A be an $n \times n$ matrix with a diagonal canonical form. How is the rank of A related to the number of zero eigenvalues of A? Show that the eigenvectors of A that correspond to non-zero eigenvalues form a basis for the column space of A.

(b) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix, where $m < n$. Can the dimension of the nullspace of BA be zero? Explain.

Problem 2. Find the Jordan canonical form J of the matrix A and a 3×3 matrix P such that $A = PJP^{-1}$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 3. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be n-component column vectors with real elements; $\mathbf{v}_1$ and $\mathbf{v}_2$ are not zero vectors and they do not lie on the same line. Find the value of scalar c that minimizes the norm of $\mathbf{u}$ where $\mathbf{u} = \mathbf{v}_1 - c\mathbf{v}_2$. Show that, for this value of c, vectors $\mathbf{u}$ and $\mathbf{v}_2$ are orthogonal to each other.

Problem 4. **(i)** Suppose $X \sim N(\theta, \sigma^2)$, and $g(x)$ is a real valued differentiable function, such that $E|g'(X)| < \infty$. Show that,

$$E\{g(X)\ (X - \theta)\} = \sigma^2\ Eg'(X).$$

Use the above to compute the *central moments* $\mu_n := E(X - \theta)^n$, where $n = 1, 2, \cdots$.

(ii) Prove that for any random variables (X, Y, Z),

$$\text{cov}\ (X\ ,\ Y) = E\{\ \text{cov}\ (X, Y \mid Z)\} +\ \text{cov}\ (E(X \mid Z)\ ,\ E(Y \mid Z)),$$

where $\text{cov}\ (X, Y \mid Z)$ is the conditional covariance of X and Y, given Z. What is the corresponding decomposition formula for $\text{var}\ (X)$ that is the basis of the Rao-Blackwell improvement of an unbiased estimator of a parameter ?

Problem 5. **(i)** Let the random vector (X_1, X_2, X_3, X_4) have the p.d.f.

$$f(x_1, x_2, x_3, x_4) = 24e^{-x_1-x_2-x_3-x_4}, \quad 0 < x_1 < x_2 < x_3 < x_4 < \infty.$$

Show that the in-between gaps (*spacings*), defined by the random variables

$$U_1 := X_1, \quad U_2 := X_2 - X_1, \quad U_3 := X_3 - X_2, \quad U_4 := X_4 - X_3$$

are statistically independent and find their distributions.

(ii) Construct the *uniformly minimum variance unbiased* (UMVU) estimator V of $\tau(\theta) := P_\theta(X = 1) = k\theta(1-\theta)^{k-1}$, based on a random sample $(X_1, X_2, \cdots, X_n)$ of size n from a binomial (k, θ) distribution. Why is it unique?

Problem 6. **(i)** Suppose we have a discrete non-negative integer valued population distribution, with probability mass function (p.m.f.) $f(x)$ $x = 0, 1, 2, \cdots$. Derive the *most powerful* (MP) randomized test of *exact* level of significance $\alpha = 0.05$, for testing H_0 vs. H_1, where

$$H_0 \; : \; f(x) = \frac{1}{2^{x+1}}, \quad x = 0, 1, 2, \cdots$$
$$H_1 \; : \; f(x) = \frac{1}{4}\left(\frac{3}{4}\right)^x, \quad x = 0, 1, 2, \cdots$$

based on a single observation. Compute the power of this test.

(ii) Can you set up the hypotheses H_0 and H_1 above as a *parametric* testing problem involving a single parameter θ? If so, then the corresponding version of the null hypothesis would be $H_0 \; : \; \theta = \theta_0$, for some arbitrary but fixed θ_0. For what composite alternatives H_1, would a uniformly most powerful (UMP) test exist, and why ?