Doctoral Qualifying Exam: Applied Mathematics

Wednesday June 14, 2006

Problem 1. In a model for the concentration C of a drug that is delivered to the bloodstream it is assumed that C decreases in time at a rate proportional to the amount that is present. Let the constant of proportionality be $\kappa > 0$. Comment on the applicability or likely validity of this model.

The drug is administered periodically, every T hours, with a dosage C_0 that is assumed to enter the bloodstream instantaneously when it is administered. Write down the mathematical formulation of this problem, and solve it to find the following:

(a) What is the concentration C_n of the drug that is present after n and before $n+1$ doses have been administered? Show that this has a periodic behavior C_∞ for large time, i.e., as $n \to \infty$, and find it.

(b) Sketch the concentration C of the drug versus time when (i) $\kappa T \gg 1$ and when (ii) $\kappa T = O(1)$. In the second case, show that for large time ($n \to \infty$) the concentration oscillates between a minimum value L and a maximum value H; show that $H - L = C_0$ and find an expression for either L or H in terms of C_0, κ, and T.

(c) If the values of κ, L, and H are known, find an expression for the value of T that must be used to maintain the concentration for large times between L and H.

Problem 2. A Sturm-Liouville operator $\mathcal{L}$ is given by

$$Lu = -\frac{d^2u}{dx^2} \quad x \in (0,l), \quad B_1 u = \frac{du}{dx}(0), \quad B_2 u = u(l),$$

with weight $w = 1$. Give the spectrum of $\mathcal{L}$, i.e., the eigenvalues and normalized eigenfunctions, explicitly.

Find the solution of the boundary value problem

$$(L - \mu)u = f(x) \quad x \in (0,l), \quad B_1 u = c_1, \quad B_2 u = c_2,$$

as an expansion in terms of eigenfunctions of the operator $\mathcal{L}$ above when the parameter μ is *not* an eigenvalue. Identify the series expansions of the contributions to the solution that are due to each of c_1, c_2, and $f(x)$ separately, and comment on the type of convergence of each series, e.g., is the convergence mean square, pointwise, or uniform, for each series. Explain your reasoning.

Problem 2 continued on next page

When the parameter $\mu = \lambda_m$ is an eigenvalue for some m, under what circumstances does a solution for u exist and what form does the solution take?

Problem 3. Let

$$Lu = -\frac{d^2u}{dx^2} \quad x \in (0,l), \quad B_1u = u(0), \quad B_2u = u(l).$$

Construct the Green's function for the differential operator $L - \mu$ where μ is a complex parameter with the same boundary operators, and show that it can be written as

$$G(x_<, x_>; \mu) = \frac{\sin\sqrt{\mu}x_< \sin\sqrt{\mu}(l - x_>)}{\sqrt{\mu}\sin\sqrt{\mu}l},$$

where $x_< = \min(x,\xi)$ and $x_> = \max(x,\xi)$.

Consider $G(x_<, x_>; \mu)$ as a function of the complex parameter μ, and find the location of its poles together with the residue at each pole. Use the result that

$$\frac{1}{2\pi i}\int_{C_\infty} G(x,\xi;\mu)d\mu = -\delta(x-\xi) = -\sum_{n\in S}\phi_n(x)\phi_n(\xi)$$

and evaluate the integral on the left of this expression where C_∞ is a circle in the μ-plane with center at the origin and radius R in the limit $R \to \infty$ and S is an index set to find:

(i) The eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of $\mathcal{L}$. You can check your answer by constructing the spectrum directly.
(ii) With the spectral representation of $\delta(x-\xi)$, form $\int_0^l f(\xi)\delta(x-\xi)d\xi$, where f is an arbitrary function, to find the corresponding transform pair. Describe the transform.

Problem 4. Solve the initial value boundary value problem

$$u_t - u_{xx} = 0, \quad x \in (0,\infty), \; t > 0$$

$$u(0,t) = 1 + b\sin\omega t, \; t > 0, \quad u(x,0) = 0, \; x > 0$$

where $b > 0$. What is the behavior of the solution as $t \to \infty$?

Problem 5. Consider the boundary value problem

$$\nabla^2 u + k^2 u = f(x,y), \quad (x,y) \in D$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0$$

where $k > 0$ is given and $D = \{(x,y) | \, 0 < y < 1, |x| < \infty\}$. The source term $f(x,y)$ has compact support, that is, $f(x,y)$ is identically zero outside a compact region Ω which is contained completely within D. The solution u also satisfies the boundary condition that it represent outgoing waves as $|x| \to \infty$.

Construct the Green's function for this problem and use it to express the solution u of the boundary value problem. Show that for x positive and sufficiently large

$$u \sim \sum_{n=0}^{M} T_n \cos n\pi y \, e^{ik_n x},$$

where $k_n = \sqrt{k^2 - n^2\pi^2} > 0$ for $n \leq M$, and express the coefficient T_n in terms of f. Explain why the sum terminates when $n = M$, and what this corresponds to physically.

Problem 6. Let G satisfy the boundary value problem

$$\nabla^2 G = -\delta(\boldsymbol{x} - \boldsymbol{x}'), \quad \boldsymbol{x}, \ \boldsymbol{x}' \in \Omega$$

$$G = 0, \quad \boldsymbol{x} \in \partial\Omega$$

where Ω is a compact region with a smooth boundary $\partial\Omega$. Physically, G represents the temperature produced by a point source at $\boldsymbol{x} = \boldsymbol{x}'$ with the boundary of Ω held at a fixed temperature.

Show that $G > 0$ in Ω and $\frac{\partial G}{\partial n} < 0$ on $\partial\Omega$, where $\frac{\partial G}{\partial n}$ denotes the outward going normal derivative. (Hint: Remove a small sphere of radius ϵ centered at $\boldsymbol{x}'$. Use the max-min theorem in the resulting multiply connected region.)