Ph.D. Qualifying Exam: Linear Algebra and Numerical Methods

Thursday June 15, 2006

Problem 1. (a) Let A be an $n \times n$ matrix with a diagonal canonical form. How is the rank of A related to the number of zero eigenvalues of A? Show that the eigenvectors of A that correspond to non-zero eigenvalues form a basis for the column space of A.

(b) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix, where $m < n$. Can the dimension of the nullspace of BA be zero? Explain.

Problem 2. Find the Jordan canonical form J of the matrix A and a 3×3 matrix P such that $A = PJP^{-1}$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 3. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be n-component column vectors with real elements; $\mathbf{v}_1$ and $\mathbf{v}_2$ are not zero vectors and they do not lie on the same line. Find the value of scalar c that minimizes the norm of $\mathbf{u}$ where $\mathbf{u} = \mathbf{v}_1 - c\mathbf{v}_2$. Show that, for this value of c, vectors $\mathbf{u}$ and $\mathbf{v}_2$ are orthogonal to each other.

Problem 4. Newton's method for finding the root of the equation $f(x) = 0$ (where f is at least twice continuously differentiable) has the following iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

(a) Suppose that α is a simple root of $f(x) = 0$. Show that

$$\alpha - x_{n+1} = -(\alpha - x_n)^2 \cdot \frac{f''(\xi_n)}{2f'(x_n)}, \quad n \geq 0$$

with ξ_n between x_n and α.

(b) Suppose now that α is a multiple root of $f(x) = 0$ with multiplicity m ($m \geq 2$). Then Newton's method is only a linear method for this case. Find the rate of convergence of Newton's method for this case.

Problem 5. The Legendre polynomials $P_n(x)$ form an orthogonal family on $[-1, 1]$ with respect to the weight $w(x) = 1$. They can be defined via the triple recursion relation

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x),$$

and $P_0(x) = 1,\ P_1(x) = x$.

(a) Show that $P_{2n}(\sqrt{1-x})$, $n \geq 0$ form an orthogonal family on $[0, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x}}$.

(b) Consider now the computation of the integral

$$I(f) = \int_0^1 \frac{f(x)}{\sqrt{1-x}} dx$$

by a two-point quadrature $I_2(f) = w_1 f(x_1) + w_2 f(x_2)$. Find the weights w_1, w_2 and the nodes x_1, x_2 so that the numerical quadrature $I_2(f)$ achieves the maximal degree of precision.

Problem 6. Consider the initial value problem:

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

(a) Consider the Trapezoidal rule for solving the initial value problem:

$$y_{n+1} = y_n + \frac{h}{2} \left(f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right).$$

Find the order of convergence of this method. Is it A-stable? Explain.

(b) The Trapezoidal rule is an implicit method and it is usually solved via a predictor-corrector scheme. If Euler's method is applied as the predictor, we end up with Heun's method:

$$y_{n+1} = y_n + \frac{h}{2} \left(f(t_n, y_n) + f(t_{n+1}, y_n + h f(t_n, y_n)) \right).$$

Find the order of convergence of this method. Is it A-stable? Explain.