Ph.D. Qualifying Examination: Distribution Theory, Statistical Inference, Linear Algebra
June 14, 2005

1. Let the matrix A be defined as follows:

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

   (a) Given that the eigenvector corresponding to the largest eigenvalue of A is of the form $[1, k, 1]^T$, find the exact value of k using the Raleigh Quotient.

   (b) Use the result of part (a) and the Rayleigh Quotient to find the remaining two eigenvalues.

2. Prove the following:

   (a) A Hermitian matrix is positive definite if and only if all of its eigenvalues are positive.

   (b) Compute e^{Kt} for the skew-hermitian matrix

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

   (a) Find B such that $J = B^{-1}AB$ is in Jordan canonical form.

   (b) Use the result in (a) to find the general solution to the equation $x' = Ax$ with $x(0) = x_0$

4. (i)$X_1, \cdots, X_n$ are i.i.d. exponentially distributed random variables with mean λ^{-1}. Show that, for $c \geq 0$,

$$P\{\min_{i \neq j,\, (i.j) \in \{1,\cdots,n\}} \mid X_i - X_j \mid \geq c\} = \exp\{-\frac{\lambda}{2} n(n-1)c\}.$$

(ii) From an urn containing a white and b black balls, a certain number of balls are removed one by one at random (without replacement), their colors being unnoted. Let A be the *event* that the next draw of a randomly chosen ball produces a white ball. Find the distribution of the indicator variable 1_A.

5. For a "*parent*" distribution with density / p.m.f. $f(x;\theta)$, satisfying the *usual regularity conditions*; suppose the likelihood $L := \prod_{i=1}^{n} f(x_i;\theta)$, based on a random sample $(X_1, \cdots, X_n)$, satisfies

$$\frac{\partial}{\partial\theta} \ln L = k(\theta)\,(T - g(\theta))\,, \quad \theta \in \Theta$$

for some *statistic* $T \equiv T(X_1, \cdots, X_n)$, and functions $g(\theta)$ and $k(\theta)$ such that $k(\theta)$ never vanishes. Then,

a) prove that T is the unique UMVU estimator of $g(\theta)$, whose variance attains the Rao-Cramer lower bound. Why is it unique?

b) use Rao-Blackwellization to get the same UMVU estimator.

6. (i) Suppose X has *Cauchy* distribution with location parameter $\theta \in (-\infty, \infty)$, with density,

$$f(x) = \frac{1}{\pi\{1 + (x-\theta)^2)\}}; \quad -\infty < x < \infty$$

Show that for testing the null hypothesis $H_0 : \theta = 0$ vs. the alternative $H_1 : \theta = 1$, using a single observation X; the *critical region* $\{1 < X < 3\}$ is the "most powerful" for its *size*. Evaluate the corresponding size and power of this test.

(ii) Consider the Beta(α, β) family of densities

$$f(x;\alpha,\beta) = \text{Constant } x^{\alpha-1}(1-x)^{\beta-1}; \quad 0 < x < 1; \quad \alpha > 0,\ \beta > 0.$$

Suppose $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_m)$ are random samples from Beta$(\mu, 1)$ and Beta$(\theta, 1)$ distributions *respectively*. Also assume that the Xs are independent of the Ys.

Show that the Likelihood Ratio (LR) test for $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$, is based on the statistic

$$T^* := \frac{\sum_i \ln X_i}{\sum_i \ln X_i + \sum_j \ln Y_j}$$

What is the null distribution of T^*?