Linear Algebra and Numerical Methods Qualifying Exam
June 14, 2005

1. Let the matrix A be defined as follows:

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

   (a) Given that the eigenvector corresponding to the largest eigenvalue of A is of the form $[1, k, 1]^T$, find the exact value of k using the Raleigh Quotient.

   (b) Use the result of part (a) and the Rayleigh Quotient to find the remaining two eigenvalues.

2. Prove the following:

   (a) A Hermitian matrix is positive definite if and only if all of its eigenvalues are positive.

   (b) Compute e^{Kt} for the skew-hermitian matrix

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

   (a) Find B such that $J = B^{-1}AB$ is in Jordan canonical form.

   (b) Use the result in (a) to find the general solution to the equation $x' = Ax$ with $x(0) = x_0$

4. Let $F_\epsilon \equiv f(x) + \epsilon g(x)$, and denote a root of F_ϵ by $\alpha(\epsilon)$. The original root α of $f(x)$ is thus $\alpha(0)$. We assume that $\alpha(0)$ is a simple root of $f(x)$ and thus $f'(\alpha(0)) \neq 0$.

(a) For sufficiently small ϵ, show that

$$\alpha(\epsilon) = \alpha(0) - \epsilon \frac{g(\alpha(0))}{f'(\alpha(0))} + O(\epsilon^2),$$

(b) Determine the fixed point and the order of convergence of the following iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}.$$

(c) Let $f(x) = (x-1)(x-2)$, $g(x) = \frac{x(x^2+3a)}{3x^2+a} - x$ and $F_\epsilon(x) = f(x) + \epsilon g(x)$. Estimate the new roots $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ (where $\alpha_1(0) = 1$ and $\alpha_2(0) = 2$) from part (a). Find the value of a so that the $\alpha = 1$ root remains unperturbed.

5. Consider the initial value problem

$$\begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0. \end{aligned}$$

The following method has been proposed as a means of numerically approximating the solution to this equation:

$$y_{n+1} = -2y_n + 3y_{n-1} + 3hf(x_n, y_n) + hf(x_{n-1}, y_{n-1}),$$

where h is the step size.

(a) What is the order of this method?

(b) Discuss the numerical stability of this method.

6. Suppose we solve the linear system $T\mathbf{x} = \mathbf{b}$ of n equations in n unknowns through the use of Jacobi's method. Recall that in Jacobi's method, we first split the matrix $T = N + P$, where N is a diagonal, non-singular matrix.

(a) What is operation count per iteration?

(b) What is the necessary and sufficient condition for convergence?

(c) Assume n is large, and that the spectral radius of the matrix P is $\sigma \sim 1 - \frac{1}{2}(\frac{\pi}{n+1})^2$ as $n \to \infty$. Write an expression for the number of iterations required to obtain an accuracy of 10^{-10} if the error after the first iteration is 1. From this determine the total number of operations as a function of n, and estimate the time it takes on a computer with a flop rate of 10^9 per second.