Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Statistics Part B: Real Analysis and Statistical Inference May 2015

The first three questions are about Real Analysis and the next three questions are about Statistical Inference.

1. Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point, and

$$|T(x) - T(y)| < |x - y| \quad \text{for all} x \neq y \in \mathbb{R}.$$

Why doesn't this example contradict the contraction mapping theorem.

2. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f. What does the Fourier series converge to? (Use complex form of the Fourier series $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$.)

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ and show that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

3. Consider

$$I = \int_0^{\infty} \frac{\log x}{x^{1/2}(1 + x^2)} f(x)\, dx.$$

where $f(x)$ is a discontinuous function that equals 1 a.e. on $[0, \infty)$.

Use Levi's monotone convergence theorem to prove that I exists as a Lebesgue integral.

4. Let $X_1, \dots X_n$ be independent random variables with density

$$f_{X_i}(x; \theta) = \frac{1}{2i\theta}, -i(\theta - 1) \leq x \leq i((\theta + 1), f_{X_i}(x; \theta) = 0, elsewhere, 0 < \theta < \infty, i = 1, 2, ..., n.$$

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Also, find the MLE of $Var| \frac{X_i}{i} - 1 |$.

(c) State and prove at least one property of the MLE of θ.

5. Let $X_1, \ldots X_n$ be independent random variables with density

$$f_{X_i}(x;\theta) = e^{i\theta - x}, i(\theta) \leq x, f_{X_i}(x;\theta) = 0, elsewhere, -\infty < \theta < \infty, i = 1, 2..., n.$$

Find the complete and sufficient statistic for θ and compute the unique minimum variance unbiased estimator of θ.

6. Let $X_n, n = 1, 2, ...$ be the sequence of random variables. This X_n, has negative binomial distribution that counts the number of failures until r_n successes have been observed, with probability of a success p_n. Here, $r_n \to \infty$ such that $r_n(1 - p_n) = \lambda_n \to \lambda(> 0)$. Find the limiting distribution of X_n as $n \to \infty$.