Department of Mathematical Sciences
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Part B: Real Analysis and Statistical Inference — Doctoral Qualifying Exam, May 2014

The first three questions are about Real Analysis and the next three questions are about Statistical Inference.

1. Let $f : \mathbb{R}^2 \to [0, \infty)$ be a measurable function with respect to the two-dimensional Lebesgue measure.

   (a) State the definition of the Lebesgue integral $\int_{\mathbb{R}^2} f(x)dx$ in terms of the measures of the level sets of f.

   (b) Now let

   $$f_\lambda(x) = e^{-\lambda(x_1-a_1)^2-\lambda(x_2-a_2)^2}, \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

   for some $a = (a_1, a_2) \in \mathbb{R}^2$ and $\lambda > 0$. Use the definition of the Lebesgue integral in (a) to compute $\int_{\mathbb{R}^2} f_\lambda(x)dx$.

   (c) Use Lebesgue dominated convergence theorem to prove that

   $$\lim_{\lambda\to\infty} \int_{\mathbb{R}^2} |f_\lambda(x)|^2 dx = 0.$$

   (d) Find the Fourier transform $\widehat{f_\lambda}$ of f_λ and show that $\widehat{f_\lambda} \to 0$ in $L^2(\mathbb{R}^2)$ as $\lambda \to \infty$.

2. Let

   $$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \qquad x \in \mathbb{R}.$$

   (a) Show that $f \in C^\infty(\mathbb{R})$, but that $f \notin C_c^\infty(\mathbb{R})$.

   (b) Show that $f \in L^2(\mathbb{R})$, but that $f \notin L^1(\mathbb{R})$.

   (c) Show that $f \in H^1(\mathbb{R})$.

   (d) What happens with the above statements, if $f(0) = 0$ instead?

3. (a) State the Sobolev embedding theorem for $H^1(\mathbb{R}^n)$.

   (b) For $L > 0$, let

   $$H_L := \{ f \in H^1(\mathbb{R}) : f(x) = 0 \ \ \forall x \in (-\infty, -L) \cup (L, +\infty)\},$$

   i.e., let H_L be a subset of $H^1(\mathbb{R})$ consisting of functions vanishing outside $[-L, L]$. Show that

   $$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{L}\, \|f'\|_{L^2(\mathbb{R})} \qquad \forall f \in H_L.$$

4. Let $X_1, \ldots X_n$ be iid random variables with density

   $$f(x; \theta) = \frac{exp\{-(x-\theta)\}}{\{1 + exp\{-(x-\theta)\}\}^2}, -\infty < x < \infty, -\infty < \theta < \infty.$$

(a) Show the existence and uniqueness of maximum likelihood estimator.

(b) Construct the iterative one-step estimator based on Newton's procedure giving the initial esimator.

5. Let $X_1, \ldots, X_n$ be a random sample from the uniform $(0, \theta]$ distribution.

(a) Find the complete sufficient statistics for θ and show how to compute the unique minimum variance unbiased estimator of

$$g(\theta) = \frac{e^{t\theta} - 1}{t\theta}, t \neq 0,$$

i.e., the mgf of the distribution.

(b) Find the close form of the unique minimum variance unbiased estimator of $g(\theta)$.

6. (a) Let X be a *single* observation from the density $f(x; \theta) = (2\theta x + 1 - \theta) I_{[0,1]}(x)$, where $-1 \leq \theta \leq 1$. Find the likelihood ratio test with significance level α of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

(b) If $X_i, \quad i = 1, \ldots, n$, are observations from the Normal distribution with known variance σ_i^2, respectively, and $X_i^{\prime}s$ are mutually independent, construct a test for testing that their means are all equal.