Department of Mathematical Sciences
New Jersey Institute of Technology

Part B: Real and Complex Analysis — Doctoral Qualifying Exam, May 2013

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Consider a differentiable function g on $\mathbb{R}$ such that $|g'(x)| \leq r < 1$. Let F_0 be bounded on $\mathbb{R}$ and define
$$F_n(x) = g(F_{n-1}(x)), \quad n \geq 1.$$
   (a) Show that $|g(u) - g(v)| \leq r|u - v|$, where u and v are any two real numbers.
   (b) Use this previous inequality with $u = F_{n-1}(x)$ and $v = 0$ to show that $|F_n(x)| \leq |g(0)| + r|F_{n-1}(x)|$, and deduce that F_n is bounded.
   (c) Using the same procedure show that $|F_{n+1}(x) - F_n(x)| \leq r^n |F_1(x) - F_0(x)|$.
   (d) Show that if $n > m$, $|F_n(x) - F_m(x)| < |F_1(x) - F_0(x)| \dfrac{r^m}{1-r}$. What conclusion can we make if $|F_1(x) - F_0(x)| \dfrac{r^m}{1-r} < \varepsilon$ with $m \geq N$.

2. Let $\phi(t) = \displaystyle\int_0^\infty e^{-xt} \frac{\sin x}{x} dx$ for $t > 0$, and $g(x,t) = \frac{\sin x}{x} e^{-xt}$.
   (a) Show that: $\phi'(t) = -1/(1+t^2)$.
   (b) Show that: $\phi(t) \to 0$ as $t \to +\infty$ using the dominated convergence theorem with $g_\nu(x) = g(x, t_\nu)$ where $1 \leq t_1 \leq t_2 \leq ...$ and $t_\nu \to +\infty$ as $\nu \to \infty$.
   (c) Show that $\phi(t) = \pi/2 - \tan^{-1} t$.
   (d) Define $\phi_\nu(t) = \int_0^\nu e^{-xt} \frac{\sin x}{x} dx$. Show that the function
$$f_\nu(t) = \begin{cases} \phi'_\nu(t) & \text{if } 0 \in\leq t \leq \nu \\ 0 & \text{if } t > \nu \end{cases}$$
   is Lebesgue integrable on $[0, +\infty)$, and find a function that dominates it.
   (e) Use dominated convergence theorem to show that $\displaystyle\lim_{\nu\to\infty} \int_0^\infty f_\nu = -\frac{\pi}{2}$, and therefore $\lim_{\nu\to\infty} \phi_\nu(0) = \pi/2$.
   (f) Use previous results to show that $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. Hint: Consider $\alpha > 0$ with $\nu = [\alpha]$ ([] means the greatest integer), and then decompose $\int_0^\nu = \int_0^\nu + \int_\nu^b$.

3. Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ and if $\int_{-\pi}^{\pi} f(x) dx = 0$, then
$$\int_{-\pi}^{\pi} |f|^2 dx \leq \int_{-\pi}^{\pi} |f'|^2 dx.$$
   Hint: Start by showing that if a_n and b_n are the Fourier coefficients of f, then $a'_n = nb_n$ and $b'_n = -na_n$ are the Fourier coefficients of f', and then use Parseval's inequality.

4. Let
$$f(z) = u + iv.$$
be an entire function such that
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \text{ in } \mathbb{C}.$$

Use the Cauchy–Riemann equations to prove that f must have the form
$$f = \alpha + \beta z,$$
where α is complex and β is real.

5. Consider the function defined as
$$f(z) := \frac{1}{z\,(z^2+4)}.$$

   (a) Describe all singularities of f in the extended complex plane, and compute the residues of each.

   (b) Find the Laurent series expansion of f for $0 < |z| < 2$.

   (c) Find the Laurent series expansion of f for $2 < |z|$.

   (d) Suppose one started with the series in (b), and wanted to extend the function by analytic continuation beyond $|z| = 2$ by expanding in a power series around $z_0 = (3/2)(1+i)$. How would you determine the coefficients of the power series, how could you be sure the disk of convergence of the power series overlaps the disk $\{z \in \mathbb{C} : |z| \leq 2\}$ - in particular, what is the radius of convergence, and how do you know that this power series truly extends the function defined by the series in (b).

6. (a) Show that the integral
$$\int_C \frac{dz}{z(z-1)},$$
where $C := \{z \in \mathbb{C} : |z| = 2\}$, is equal to zero, using both (i) the Residue Theorem and (ii) Rouché's Theorem. (Hint: Use the function $f(z) = z/(z - 1)$).

   (b) Use the Residue Theorem, verifying all details, to evaluate
$$\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+4)}.$$