DEPARTMENT OF MATHEMATICAL SCIENCES
New Jersey Institute of Technology

Part C: Linear Algebra and Numerical Methods — DOCTORAL QUALIFYING EXAM, MAY 2013

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. Let $P_{C(A)}$ denote the orthogonal projection operator onto the column space of matrix A.
   (a) Show that
   $$UP_{C(A)} = P_{C(UA)}U$$
   for any real $m \times n$ matrix A and unitary $m \times m$ matrix U.
   (b) Show that $P_{C(A)} + P_{C(B)}$ is a projection if and only if $P_{C(A)}P_{C(B)} = 0$.

2. Let $V \subset \mathbb{R}^3$ be the subspace spanned by $(4, 1, -2)^T$. Suppose we define an inner product for $x, y \in \mathbb{R}^3$ by
   $$(x, y) = 2x_1y_1 + 3x_2y_2 + x_3y_3.$$
   (a) Find a basis for $V^\perp = \{x|(x, y) = 0 \; \forall y \in V\}$.
   (b) Find C and D in the linear fit $y(t) = Ct + D$ that minimizes the weighted squared error $||y(t) - (Ct + D)||^2$, as defined using the norm induced by the above inner product, for data $y(-1) = -1$, $y(0) = 0$, $y(1) = 0$.

3. Consider the system of differential equations given by
   $$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} x.$$
   (a) Find the fundamental solution $X(t)$ satisfying $X(0) = I$.
   (b) Show that if $t \geq 0$,
   $$||X(t)|| = \left(1 + \frac{1}{2}t^2 + \frac{1}{2}t\sqrt{4 + t^2}\right)^{1/2} e^t$$
   where $||X||$ is the standard matrix (operator) norm
   $$||X|| = \max_{x \neq 0} \frac{||Xx||}{||x||}$$
   and $||x||$ is the Euclidean norm of vector x.

4. Explain how Newton's method can be used to find the solutions to the equation $x^2 = 2$. Write down the approximate solution after two iterations with initial guess $x_0 = 1$. Show that the Newton's iterations converge for any nonzero initial guess.

5. Consider the initial value problem

$$y' = f(t,y), \;\; y(0) = y_0.$$

One possibility to derive numerical schemes for the initial value problem above is to consider the following family of k-multistep methods of the form

$$\sum_{j=-1}^{k-1} a_j y_{n-j} = h f_{n+1}$$

where the coefficients a_j are determined by (1) constructing the polyonomial p_k of degree k that interpolates y_{n-j} at $t_{n-j} = (n-j)h$ for $j = -1, \ldots, k-1$, (2) evaluating $p_k'(t_{n+1}) \approx y(t_{n+1})$, and (3) equating $p_k'(t_{n+1}) = f_{n+1}$.

(a) Show that for $k = 1, 2$ these methods are given explicitly by

$$\begin{aligned} y_{n+1} - y_n &= h f_{n+1} \\ (3y_{n+1} - 4y_n + y_{n-1})/2 &= h f_{n+1} \end{aligned} \tag{1}$$

(b) What is the order of these methods for $k = 1, 2$? How about for a general k?

(c) Show that for $k = 1$ and 2 the methods are stable, and for $k = 1$ the method is A-stable. Are the methods convergent in those cases?

6. (a) Consider the inner product

$$\langle f, g \rangle = \int_{-1}^{1} (x+1) f(x) g(x) dx$$

and let $\mathcal{P}_n$ be the set of polynomials of degree at most n. Let $Q \in \mathcal{P}_n$ be a nontrivial polynomial which is $\langle \cdot, \cdot \rangle$-orthogonal to $\mathcal{P}_{n-1}$. Construct such a polynomials in terms of the Legendre polynomials $\{P_0, P_1, \ldots, P_n, \ldots\}$ which are orthogonal in $L^2(-1,1])$ and normalized such that $P_n(1) = 1$ for all $n \geq 0$.

(b) Consider the quadrature

$$I_n^1(g) = \sum_{i=1}^{n} c_i g(x_i) \approx \int_{-1}^{1} (x+1) g(x) dx$$

where $\{x_i\}$ is the set of zeros of Q and c_i are chosen so that I_n^1 is exact for $\mathcal{P}_{n-1}$. Show that I_n^1 is, in fact, exact for $\mathcal{P}_{2n-1}$.

(c) Show that there exist coefficients $\{a_i\}$ for $i = 0, 1, \ldots$ which make the formula exact for $\mathcal{P}_{2n}$

$$I_n(f) = a_0 f(-1) + \sum_{i=1}^{n} a_i f(x_i) \approx \int_{-1}^{1} f(x) dx.$$

Here the nodes x_i are given in Part (b). Show that this quadrature is not exact for $\mathcal{P}_{2n+1}$.

(d) For the Gaussian quadrature rule $\int_{-1}^{1} f(x)dx \approx J_n f = \sum_{i=1}^{n} w_i f(y_i)$, prove that $\lim_{n\to\infty} J_n f = \int_{-1}^{1} f(x)dx$ for all $f \in C([-1,1])$.