Preliminary Exam in Real Analysis and Statistical Inference: Spring 2012

1. Compute the following limits. Justify the calculation with an appropriate convergence theorem, when necessary.

   - $\lim_{n\to\infty} \int_0^\infty (1+(x/n))^{-n} \sin(x/n)\; dx$

   - $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\; dx$. (The answer depends on whether $a > 0$, $a = 0$ or $a < 0$. How does this accord with the various convergence theorems?)

2. Consider the iteration scheme

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$

   with initial iterate $x_0 > 0$ corresponding to a map $T : (0,\infty) \to (0,\infty)$ given by

$$Tx = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

   - Find the fixed points of T.

   - Show that T is a contraction map on an interval of the form $[b,\infty)$ with $b > 0$. What is one possible choice for b?

   - Show that in practice the iteration converges for any starting guess $x_0 \in (0,\infty)$.

3. In this problem assume all functions are continuous on $[a,b]$. Let $\{\phi_0, \phi_1, \ldots\}$ be an orthonormal system on $[a,b]$. Prove the following three statements are equivalent:

   - $(f,\phi_n) = (g,\phi_n)$ for all n implies $f = g$. (Two continuous functions with the same Fourier coefficients are equal.)

   - $(f,\phi_n) = 0$ for all n implies $f = 0$.

   - If T is an orthonormal set on $[a,b]$ such that $\{\phi_0, \phi_1, \ldots\} \subseteq T$, then $\{\phi_0, \phi_1, \ldots\} = T$. (The orthonormal set cannot be enlarged, i.e., it is complete.)

   (Over, please)

4. Suppose that we observe $(\delta_1, X_1), \ldots, (\delta_n, X_n)$, where the data are n iid replicates of (X, δ). Here X is a positive random variable and δ is binary. The conditional distribution of δ given $X = x$ is Bernoulli with success probability $m(x, \theta)$, where $\theta \in R^k$ is an unknown parameter vector to be estimated from the observed data. Note that the likelihood function for θ can be written down in the standard way as for samples from a Bernoulli distribution but here the success probability is $m(x, \theta)$.

- Write $D_r(m(x,\theta)) = \partial m(x,\theta)/\partial\theta_r$. Show that $\sigma_{r,s}$, the (r, s) element of the Fisher information matrix $I(\theta)$, is given by

$$\sigma_{r,s} = E\left(\frac{D_r(m(X,\theta))D_s(m(X,\theta))}{m(X,\theta)(1-m(X,\theta))}\right), 1 \le r \le k, 1 \le s \le k.$$

- Describe completely the asymptotic distribution of $\hat{\theta}$, the MLE, when $k = 1$ and

$$\log\left(\frac{m(x,\theta)}{1-m(x,\theta)}\right) = \theta x.$$

  That is, name the asymptotic distribution and specify its mean and the variance.

5. To obtain the MVUE of a parameter $h(\theta)$, it is well known that one first identifies a complete sufficient statistic Y with density function $g(y;\theta)$. Then one constructs a function of Y, say $u(Y)$, such that $E[u(Y)] = h(\theta)$.

   (i) Suppose that $X_1, \ldots, X_n$ denote a random sample from a distribution with probability density function $f(x;\theta) = Q(\theta)M(x)$, $\theta \le x < b$. Show that $Y_1 = \min(X_1, \ldots, X_n)$ is a sufficient statistic for θ and hence that

$$u(y) = h(y) - \frac{h'(y)}{nQ(y)M(y)}.$$

   (ii) Suppose that $f(x;\theta) = e^{-(x-\theta)}, x \ge \theta$. Use part (i) to find the MVUE of $P(1 < X < 2)$.

6. This problem has two independent parts.

- Let X denote the number of heads when a coin is flipped independently three times. Suppose that the probability comes up heads on a single flip is θ and it is the same for each flip. To test the null hypothesis $H_0 : \theta = 1/2$ against $H_1 : \theta = 3/4$, obtain the test that minimizes the sum of type I and type II error probabilities. What are the two error probabilities for the test?
- Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from the normal populations $N(\mu_1, 4\sigma^2)$ and $N(\mu_2, \sigma^2)$ respectively. Develop the likelihood ratio test for testing the null hypothesis $H_0 : \mu_1 = \mu_2$ versus the alternative $H_1 : \mu_1 \neq \mu_2$.