Preliminary Exam in Real and Complex Analysis: Spring 2012

1. Compute the following limits. Justify the calculation with an appropriate convergence theorem, when necessary.

   - $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\ dx$

   - $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\ dx$. (The answer depends on whether $a > 0$, $a = 0$ or $a < 0$. How does this accord with the various convergence theorems?)

2. Consider the iteration scheme
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$
with initial iterate $x_0 > 0$ corresponding to a map $T : (0, \infty) \to (0, \infty)$ given by
$$Tx = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

   - Find the fixed points of T.

   - Show that T is a contraction map on an interval of the form $[b, \infty)$ with $b > 0$. What is one possible choice for b?

   - Show that in practice the iteration converges for any starting guess $x_0 \in (0, \infty)$.

3. In this problem assume all functions are continuous on $[a, b]$. Let $\{\phi_0, \phi_1, \ldots\}$ be an orthonormal system on $[a, b]$. Prove the following three statements are equivalent:

   - $(f, \phi_n) = (g, \phi_n)$ for all n implies $f = g$. (Two continuous functions with the same Fourier coefficients are equal.)

   - $(f, \phi_n) = 0$ for all n implies $f = 0$.

   - If T is an orthonormal set on $[a, b]$ such that $\{\phi_0, \phi_1, \ldots\} \subseteq T$, then $\{\phi_0, \phi_1, \ldots\} = T$. (The orthonormal set cannot be enlarged, i.e., it is complete.)

4. Let f be analytic in the upper half-plane $U := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, C^1 in an open neighborhood of the closure $\bar{U}$ of this half-plane and be such that $f(x + i0)$ is real for all real x. Explain using only the Cauchy-Riemann equations and basic analytic continuation results (and not the Schwarz reflection principle) why
$$g(z) := \overline{f(\bar{z})}$$
defined in a neighborhood of the closure of the lower half-plane $L := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ is the analytic continuation of f to an entire function.

(Over, please)

5. Expand the function

$$f(z) := \frac{z+1}{z^3 - 3z^2 + 2z}$$

in powers of z valid for (a) $1 < |z| < 2$ and (b) $2 < |z|$; and (c) powers of $z+1$ valid for $3 < |z+1|$.

6. Suppose that f is analytic in the upper half-plane U above, continuous in $\overline{U}$ and there exist positive numbers α, R, M such that $|f(z)| \leq M\,|z|^{-\alpha}$ whenever $R \leq |z|$. Prove that f can be expressed in terms of only its restriction to the real axis, and explain how this representation provides solutions to two boundary-value problems for partial differential equations on U.