Preliminary Exam in Linear Algebra and Numerical Methods: Spring 2012

1. Let $T : X \to Y$ be a linear map from a vector space X to a vector space Y.

   - Show that the *kernel* (*null space*) of T, $\ker T := \{x \in X : T(x) = 0\}$, and *image* (*range*) of T, $\mathrm{im}T := \{T(x) : x \in X\}$, are subspaces of Y and Y, respectively.
   - Starting with a basis $\{x_1, \ldots, x_k\}$ for $\ker T$, prove the dimension theorem; namely, if $\dim X < \infty$, then
   $$\dim X = \dim \ker T + \dim \mathrm{im}T.$$
   - Prove that if $\dim X = \dim Y < \infty$, then the linear map T is injective (one-one) if and only if it is surjective (onto).

2. Find a simple formula for A^n for every positive integer n, where
$$A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Use the appropriate spectral theorems to prove each of the following:

   - Let U be an $n \times n$ complex matrix that is unitary, i.e. $UU^* = U^*U = I$, where U^* is the conjugate transpose of U. Show that U has a logarithm, i.e. there exists a matrix V such that $e^V = U \iff V = \log U$.
   - Let A be a complex $n \times n$ matrix such that $A = A^*$ (i.e., the matrix is Hermitian), and let $\|A\|$ be that standard matrix norm induced by the standard inner product; namely,
   $$\|A\| := \max_{|x|=1} |Ax| = \max_{|x|=1} \sqrt{(Ax, Ax)} = \max_{|x|=1} \sqrt{(Ax)^T \overline{(Ax)}}.$$
   Show that all eigenvalues λ of A satisfy $|\lambda| \leq \|A\|$ and that one or both of $+\|A\|$ or $-\|A\|$ is an eigenvalue of A.

4. The least squares approximation of an arbitrary polynomial $f(x)$ of degree $\leq n$ can be written as $r_n^*(x) = \sum_{j=0}^{n} a_j \phi_j(x)$, where $\phi_m(x), m \geq 0$, is an orthonormal family of polynomials with weight function $w(x) \geq 0$. Find $a_j, j = 0, \ldots, n$.
Hint: Define the inner product of two continuous functions f and g by
$$(f, g) = \int_a^b w(x) f(x) g(x) dx, \quad f, g \in C[\text{a,b}],$$
and the two norm by
$$||f||_2 = \sqrt{\int_a^b w(x)[f(x)]^2 dx}.$$
Solve the least squares problem by minimizing $||f - r||_2^2$, where $r(x) = \sum_{j=0}^{n} a_j \phi_j(x)$.

(Over, please)

5. Consider the trapezoidal method,

$$y(t_{i+1}) = y(t_i) + \frac{h}{2}(f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1})))$$

solved with one iteration using Euler's method,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)), i \geq 1,$$

as the predictor, for solving the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

- Form the difference equation.
- Compute the real part of the region of absolute stability for the method. (*Hint:* Use the model problem, $y' = \lambda y$ with Re(λ)< 0 to find the values of λh for which the method is stable.

6. (i) Consider the Jacobi

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(k-1)} \right], \quad i = 1, 2, \cdots, n$$

and Gauss-Seidel

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right], \quad i = 1, 2, \cdots, n$$

iterative methods for solving the linear system of equations $\mathbf{Ax} = \mathbf{b}$. Represent the Jacobi and Gauss-Seidel iterations by $\mathbf{x}^{(k)} = \mathbf{T}\mathbf{x}^{(k-1)} + \mathbf{c}$ and show that $\mathbf{T} = \mathbf{D}^{-1}(-\mathbf{L} - \mathbf{U})$ and $\mathbf{c} = \mathbf{D}^{-1}\mathbf{b}$ for the Jacobi method and $\mathbf{T} = (\mathbf{D} + \mathbf{L})^{-1}(-\mathbf{U})$ and $\mathbf{c} = (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b}$ for the Gauss-Seidel method, where $\mathbf{D}$ is a diagonal matrix whose diagonal entries are those of $\mathbf{A}$, $\mathbf{L}$ is the strictly lower-triangular part of $\mathbf{A}$ and $\mathbf{U}$ is the strictly upper-triangular part of $\mathbf{A}$.

(ii) Show if $\|\mathbf{T}\| < 1$, the Jacobi and Gauss-Seidel techniques converge and the following error bound holds

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|\mathbf{T}\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|, \quad \mathbf{x} \in \mathbb{R}^n,$$

for any matrix norm.