Ph.D. Prelim : Exam. C
Real Analysis and Statistical Inference

June 3, 2011

1. Prove that if g is of bounded variation on $[0, \delta]$, then

$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t} \, dt = g(0+).$$

2. Use Holder's Inequality to prove Minkowski's Inequality: If f and h belong to L_p, $p \geq 1$, then $f + h$ belongs to L_p and $||f + h||_p \leq ||f||_p + ||h||_p$.

3. Consider a measure space $(X, \mathbf{X}, \mu)$ and a sequence of functions f_n defined on the space.

(a) Prove or give a counterexample with explanation: If f_n converges in measure, then it converges almost everywhere.

(b) Prove or give a counterexample with explanation: If $\mu(X) < \infty$ and if $f_n \to f$ almost everywhere, then $f_n \to f$ in measure.

4. Let $\overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$ and $s^2 := \frac{1}{n-1} \sum_{i=1}^n \left(X_i - \overline{X}\right)^2$ be the mean and variance respectively, of $n \geq 2$ observations $X_1, \cdots, X_n$ which are i.d. (*identically distributed*), but need not be i.i.d., and have a finite fourth moment. Let $\theta_1 := EX_i$ and $\theta_j := E(X_i - \theta_1)^j$, $i = 1, 2, \cdots, n;\ j = 2, 3, 4$ denote the *common* mean and the second, third and fourth central moments.

(i) Show that the sample variance can be expressed as

$$s^2 = \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (X_i - X_j)^2.$$

(ii) By using the random variables $A_i := (X_i - \theta_1);\ i = 1, 2, \cdots, n$, show that,

$$Es^2 = \theta_2, \text{ and } \operatorname{var}(s^2) = \frac{1}{n} \left((\theta_4 - \theta_2^2)^2 - \frac{1}{n} \sum_{i \neq j} \operatorname{cov}(A_i^2, A_j^2) \right)$$

(iii) Compute cov $(\overline{X}, s^2)$, if it is additionally assumed that all pairs (X_i, X_j) are also jointly identically distributed. Use this to find necessary and sufficient conditions for cov $(\overline{X}, s^2) = 0$ to hold.

(iv) If the observations $(X_1, X_2, \cdots, X_n)$ described above can be regarded as a random sample from a distribution in L_4 with mean μ and variance σ^2; then using (ii), prove that the sample varince is an unbiased and consistent estimator of σ^2.

5. Based on a random sample $X_1, \cdots, X_n$ of size n, from the exponential density $f(x;\theta) = \theta^{-1}\exp\{-x/\theta\};\ x > 0,\ \theta > 0$; consider the problem of testing the null hypothesis $H_0 : \theta = \theta_0$ vs. the alternatative hypotheses $H_1 : \theta \neq \theta_0$, where θ_0 is a fixed value in the parameter space. Answer the following questions.

(a) Derive the large sample size-α *Wald*-test, *Scores*-test and the test based on $(-2\ln\Lambda)$. For each test, derive the respective *test-statistics* explicitly, and the corresponding critical regions.

(b) Derive the *exact* likelihood ratio test to show that it rejects H_0 at level of significance α iff,

$$\left\{\frac{2}{\theta_0}\sum_{i=1}^{n} X_i \leq c_1 := \chi^2_{1-\alpha/2}(2n),\ \text{ or }\ \geq c_2 := \chi^2_{\alpha/2}(2n)\right\}.$$

(c) Show that the power function of the exact likelihood ratio test in part (b) above, is

$$\gamma(\theta) = 1 - \int_{c_1\frac{\theta_0}{\theta}}^{c_2\frac{\theta_0}{\theta}} \frac{1}{2^n\Gamma(2n)} e^{-(x/2)} x^{n-1} dx.$$

6. Consider a random sample $X_1, \cdots, X_n$ of size n from the Normal distribution $N(\mu, 1)$, where the mean μ is unknown. The parameter of interest is $g(\mu) := P(X_1 < 0)$.

(a) Show that the unique UMVU estimator of $g(\mu)$ is $\Phi\left(-\sqrt{\frac{n}{n-1}}\,\overline{X}\right)$.

(b) What can you say about the difference of the UMVU estimator and the maximum lilelihood estimator of $g(\mu)$, as the sample size $n \to \infty$?