Preliminary Exam in Real and Complex Analysis: Spring 2011

1. Prove that if g is of bounded variation on $[0, \delta]$, then

$$\lim_{\alpha\to\infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t}\, dt = g(0+).$$

2. Use Holder's Inequality to prove Minkowski's Inequality: If f and h belong to L_p, $p \geq 1$, then $f + h$ belongs to L_p and $||f + h||_p \leq ||f||_p + ||h||_p$.

3. Consider a measure space $(X, \mathbf{X}, \mu)$ and a sequence of functions f_n defined on the space.

   (a) Prove or give a counterexample with explanation: If f_n converges in measure, then it converges almost everywhere.

   (b) Prove or give a counterexample with explanation: If $\mu(X) < \infty$ and if $f_n \to f$ almost everywhere, then $f_n \to f$ in measure.

4. (a) Replace the integrand with an appropriate complex function and use integration around a closed semicircular contour indented at the origin to prove that (assume $p > 0$)

$$\int_0^\infty \frac{\sin(px)}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} \left[1 - e^{-ap}\right].$$

   (b) Prove the following integration result (assume p and C are real and $C > 0$) by analytically extending the integrand into the complex plane and integrating over a rectangular contour with corners at $\pm R$, $\pm R + i2\pi$. When solving this problem, make sure to obtain the condition(s) on the values of p required for convergence of this integral.

$$\int_{-\infty}^\infty \frac{e^{px}}{C + e^x}\, dx = \frac{\pi C^{p-1}}{\sin \pi p}.$$

5. If $f(z)$ has a pole of order m at $z = z_0$, then its residue can be calculated as

$$Res(f; z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

   (a) Show that the above expression follows from the Cauchy Integral Formula (which relates the value of an analytic function and its derivatives at a given point with a certain contour integral around that point).

   (b) Show that even though $f(z) = z/\sinh^2 z$ has a simple pole at $z = 0$ (find a couple terms in its Laurent series about $z = 0$), the above expression remains valid even when "incorrectly" setting $m = 2$ instead of $m = 1$ in this formula.

(c) Prove that the above residue calculation expression remains valid for any integer m which is greater or equal to the order of the pole, k (hint: use series representation and/or the Cauchy Integral Formula).

6. The Maximum Modulus Principle states that if $f(z)$ is a non-constant function analytic in D, then $|f(z)|$ cannot attain its maximum in the interior of D.

   Suppose $f(z)$ is a non-constant function analytic in domain D. Prove the following statements:

   (a) Using only the Maximum Modulus Principle above, prove that $|f(z)|$ cannot attain a *non-zero* minimum within domain D. Conversely, show (by giving one example) that $|f(z)|$ *may* in fact attain a *zero* minimum within domain D.

   (b) Show that neither maxima *nor* minima of a non-constant *real* function harmonic in domain D can be attained within D. To do this, apply the Maximum Modulus Principle to function $e^{f(z)}$, where $f(z)$ is a non-constant analytic function.