# Preliminary Exam in Applied Mathematics: June 2011

1. The SIR epidemic model is a simple mathematical model that governs the spread of an infectious disease in a fixed population of $\hat{N}$ individuals. Let $\hat{S}(t)$ denote the number of individuals who are susceptible to the disease, $\hat{I}(t)$ the number of infected individuals and $\hat{R}(t)$ the number who have recovered, and are no longer susceptible. The governing equations may be written:

$$\frac{d\hat{S}}{d\hat{t}} = -\beta \hat{S}\hat{I}, \qquad \frac{d\hat{I}}{d\hat{t}} = \beta \hat{S}\hat{I} - \gamma \hat{I}, \qquad \frac{d\hat{R}}{d\hat{t}} = \gamma \hat{I},$$

with initial conditions

$$\hat{I}(0) = \hat{I}_0, \quad \hat{R}(0) = 0, \quad \hat{S}(0) = \hat{N} - \hat{I}_0.$$

   (a) Sum the three differential equations above and solve for the quantity $\hat{S} + \hat{I} + \hat{R}$.

   (b) What are the units of the parameter $\gamma$? Non-dimensionalize the above equations (and scale all three dependent variables on the total population, $\hat{N}$, e.g. $I = \hat{I}/\hat{N}$).

   (c) Suppose the initial infected population is small, $I(0) = \epsilon \ll 1$. Seek a solution of your equations in the form of a power series expansion in $\epsilon$, i.e. $I = \epsilon I_1 + \cdots$, $S = 1 + \epsilon S_1 + \cdots$. Solve for $I_1$. Under what conditions will the infected population grow?

2. A linearized form of the Cahn-Hilliard equation governing the separation of two components of a fluid to form localized pure regions of each component is given by:

$$\frac{\partial C}{\partial t} = -\frac{\partial^2}{\partial x^2}\left(C + 4\frac{\partial^2 C}{\partial x^2}\right)$$

where $C(x,t)$ is the concentration of the fluid. Note that $C = 0$ is a solution to this equation. Perform a linear stability analysis of this solution by considering perturbations of the form:

$$C(x,t) = e^{\omega t}\cos(kx).$$

Determine the critical wavenumber, $k_c$, below which the solution is unstable. Determine the wavenumber, $k_{\max}$, corresponding to the fastest growing disturbance (i.e. maximum growth rate).

3. Use the general balance law to derive the partial differential equation describing traffic flow. Let $\rho$ denote the traffic density (the number of cars per mile), and $Q$ the flux of traffic (the average number of cars that pass a fixed location per hour). Suppose on a certain highway the flux is given by

$$Q(\rho) = a\rho \ln[\rho_{\max}/\rho].$$

Write down the partial differential equation for $\rho$. Now suppose the initial density, $\rho(x, 0)$, varies linearly from bumper-to-bumper traffic behind $x = -x_0$ to no traffic ahead of $x = 0$. Solve for $\rho$ using the method of characteristics. Sketch the characteristics in the $t - x$ plane, and sketch $\rho$ as a function of $x$ at both $t = 0$ and a later time.

4. Find the general solution to the following differential equations. For part (a) it is sufficient to write down the particular solution using the method of undetermined coefficients without determining the coefficient(s).
(a)

$$y''' - 3y'' + 3y' - y = \cosh(t), \tag{1}$$

(b)

$$x^2y'' - 3xy' + 4y = x^2. \tag{2}$$

5. (a) Find the general solution to the differential equtaion

$$y' = r\left(1 - \frac{y}{K}\right)y. \tag{3}$$

Explain why the solution always approaches $y = K$ if $y(0) \neq 0$. What happens if $y(0) = 0$?
(b) Find the general solution to the differential equation

$$y' + p(t)y = 0, \quad y(0) = 1, \tag{4}$$

where $p(t) = 1$ for $0 \leq t \leq 1$ and $p(t) = 2$ for $t > 1$.

6. Find all the critical points of the nonlinear system

$$\dot{y}_1 = y_1^2 - y_1y_2 - y_1, \tag{5}$$
$$\dot{y}_2 = y_2^2 + y_1y_2 - 2y_2. \tag{6}$$

Investigate the stability of all critical points. Draw the phase portrait for this nonlinear system.