Preliminary Exam in Linear Algebra and Numerical Methods: Spring 2011

1. Consider

$$A = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

   (a) Find the four fundamental subspaces (row space, column space, nullspace and left nullspace) of A.

   (b) Find the singular value decomposition for A.

   (c) Find the minimum length least-squares solution to $Ax = b$ if

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

2. A predator-prey model leads to a population dynamics given by

$$\frac{du}{dt} = Au \qquad \text{where} \qquad A = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -3/2 \end{pmatrix}. \tag{1}$$

   (a) Let $||u||^2 = u^T u$. Show using Eqn. 1 that $\frac{d}{dt}||u||^2 < 0$ if $||u|| > 0$.

   (b) Solve the above system for arbitrary initial condition u_0.

   (c) Let $w(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T u = u_1(t) + u_2(t)$. Find the initial condition u_0 that maximizes $\frac{dw}{dt}|_{t=0}$ under the constraint that $||u_0||^2 = 1$.

3. Suppose $AM + M^H A = -I$ where A is positive definite. Show that all eigenvalues of M have negative real part.

4. Let $f \in C[a, b])$. Solve the least square problem

$$||f - r||_2^2 = \int_a^b w(x) \left[f(x) - \sum_{j=0}^{n} b_j \phi_j(x) \right]^2 dx = G(b_0, ..., b_n)$$

by minimizing G, where $\{\phi_n(x), n \geq 0\}$ is an orthonormal family of polynomials with weight function $w(x) \geq 0$. Hint: compute the coefficients b_j.

5. Consider

$$\begin{cases} Y'(x) = f(x, Y(x)) \\ Y(x_0) = Y_0 \end{cases}$$

By integrating over $[x_n, x_{n+1}]$, we obtain

$$Y(x_{n+1}) = Y(x_n) + \int_{x_n}^{x_{n+1}} f(x, Y(x)) dx.$$

(a) Use the linear polynomial $p_1(x)$ interpolating $f(x, Y(x))$ at $[x_n, x_{n-1}]$ to show that

$$\int_{x_n}^{x_{n+1}} f(x, Y(x))dx \approx \int_{x_n}^{x_{n+1}} p_1(x)dx = \frac{3h}{2} f(x_n, Y(x_n)) - \frac{h}{2} f(x_{n-1}, Y(x_{n-1})).$$

where $h = x_{j+1} - x_j$ and $j = 0, .., N$.

(b) Does the resulting numerical method satisfy the strong root condition?

6. Assume the matrix A is invertible. Consider the following iterative procedure

$$\begin{aligned} Az_1^{(m+1)} &= b_1 - Bz_3^{(m)} \\ Az_2^{(m+1)} &= b_2 - Bz_3^{(m)} \\ z_3^{(m+1)} &= C(z_1^{(m+1)} + z_2^{(m+1)}) \end{aligned}$$

a) Give a sufficient and necessary condition that guarantees the convergence of this iterative procedure.

b) Give a sufficient and necessary condition that guarantees the convergence of the previous iterative procedure with

$$\tilde{z}_3^{(m+1)} = \omega z_3^{(m+1)} + (1 - \omega) z_3^{(m)}$$

where $0 \leq \omega \leq 1$.

c) Do the two previous iterative methods converge in the case where

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad C = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/2 \end{bmatrix}.$$