PhD Program in Applied Probability and Statistics
Qualifying Exam A: Real Analysis and Probability

May 27, 2010

1. (a) Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Prove that if both f and g are Riemann integrable, so is their product fg.

   (b) Is the analog of (a) true for Lebesgue integrable functions on sets of finite measure? In particular, prove that $f, g \in L^1(E)$, where E is a Lebesgue measurable subset of $\mathbb{R}$ of finite measure, implies $fg \in L^1(E)$, or produce a counterexample.

2. (a) Consider the Fourier cosine expansion of the function $f(x) = \pi - x$ on the interval $[0, \pi]$ in the usual form

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

   Determine whether or not the series converges pointwise to f, uniformly to f, and if the Fourier series can be differentiated term-by-term. Explain your answers.

   (b) Let X be a separable Hilbert space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$ and let $l^2 = l^2(\mathbb{R})$ be the standard Hilbert space of square-summable sequences $\hat{x} = (x_1, x_2, \ldots, x_k, \ldots)$ with its usual inner product

$$\langle \hat{x}, \hat{y} \rangle_0 = \sum_{n=1}^{\infty} x_n y_n.$$

   One can use a basic result on the complete orthonormal sets of X to show that X is equivalent to l^2 in a very strong sense.State and prove this result.

3. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be measurable with respect to Lebesgue measure, and suppose that $f \in L^1(\mathbb{R}^2)$. Why is the function $f^y(x) = f(x, y)$ in $L^1(\mathbb{R})$ for almost all $y \in \mathbb{R}$?

(b) Suppose in addition that $\partial_y f(x,y)$ is defined for all $y \in \mathbb{R}$ and there exists $g \in L^1(\mathbb{R})$ such that $|\partial_y f(x,y)| \leq g(x)$ for all $x \in \mathbb{R}$. Prove that the function defined as

$$F(y) = \int_{\mathbb{R}} f(x,y)dx$$

is differentiable, and

$$F'(y) = \int_{\mathbb{R}} \partial_y f(x,y)dx.$$

4. Let $\{Y_n \ : \ n = 1, 2, \cdots\}$ be an arbitrary sequence of r.v.s in L_p, and let $S_n := \sum_{i=1}^{n} Y_i$ denote their partial sums and $X_n := \frac{S_n}{n}$. Prove the following conditions for the original sequence Y_n to obey the 'weak law of large numbers' (WLLN) with centering constants $a_n \equiv 0$ and norming constants $b_n = n$:

   (a) (sufficient condition) As $n \to \infty$, $E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) \to 0$, for *some* $p > 0$.

   (b) (necessary condition) As $n \to \infty$, $E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) \to 0$, for *all* $p > 0$; and then $\left(\frac{|X_n|^p}{1+|X_n|^p}\right) \xrightarrow{P} 0$.

5. (a) Show that $X_n \xrightarrow{\text{a.s.}} X$ implies $X_n \Rightarrow X$. ($\Rightarrow$ denotes *convergence in distribution*).

   (b) Show that $X_n \xrightarrow{\text{a.s.}} X$ if and only if $P(|X_n - X| \geq \epsilon, \text{ i.o. }) = 0$, for all $\epsilon > 0$.

6. Suppose $\{X_n, \ n \geq 1\}$ are i.i.d. with $N(0,1)$ (standard Normal) as the common distribution. Using the fact that for any r.v. Z distributed as $N(0,1)$, we have $P(Z > x) \sim \phi(x)/x$, in the sense that

$$\lim_{x\to\infty} \frac{P(Z > x)}{\phi(x)/x} = 1,$$

(where $\phi(x)$ is the standard Normal density function); show that,

$$P\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\ln n}} = \sqrt{2}\right) = 1.$$