PhD Program in Applied Mathematics
Qualifying Exam A: Real and Complex Analysis

May 27, 2010

1. (a) Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Prove that if both f and g are Riemann integrable, so is their product fg.

   (b) Is the analog of (a) true for Lebesgue integrable functions on sets of finite measure? In particular, prove that $f, g \in L^1(E)$, where E is a Lebesgue measurable subset of $\mathbb{R}$ of finite measure, implies $fg \in L^1(E)$, or produce a counterexample.

2. (a) Consider the Fourier cosine expansion of the function $f(x) = \pi - x$ on the interval $[0, \pi]$ in the usual form

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

   Determine whether or not the series converges pointwise to f, uniformly to f, and if the Fourier series can be differentiated term-by-term. Explain your answers.

   (b) Let X be a separable Hilbert space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$ and let $l^2 = l^2(\mathbb{R})$ be the standard Hilbert space of square-summable sequences $\hat{x} = (x_1, x_2, \ldots, x_k, \ldots)$ with its usual inner product

$$\langle \hat{x}, \hat{y} \rangle_0 = \sum_{n=1}^{\infty} x_n y_n.$$

   One can use a basic result on the complete orthonormal sets of X to show that X is equivalent to l^2 in a very strong sense.State and prove this result.

3. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be measurable with respect to Lebesgue measure, and suppose that $f \in L^1(\mathbb{R}^2)$. Why is the function $f^y(x) = f(x, y)$ in $L^1(\mathbb{R})$ for almost all $y \in \mathbb{R}$?

(b) Suppose in addition that $\partial_y f(x,y)$ is defined for all $y \in \mathbb{R}$ and there exists $g \in L^1(\mathbb{R})$ such that $|\partial_y f(x,y)| \leq g(x)$ for all $x \in \mathbb{R}$. Prove that the function defined as

$$F(y) = \int_{\mathbb{R}} f(x,y)dx$$

is differentiable, and

$$F'(y) = \int_{\mathbb{R}} \partial_y f(x,y)dx.$$

4. (a) Consider integrating $f(z) = e^{-z^2}$ around the rectangular contour C_R with corners at $\pm R$, $\pm R + ia$. By showing that the integrals along the shorter side portions (of length a) go to zero as $R \to \infty$, deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(2ax)\, dx = \sqrt{\pi} e^{-a^2}.$$

You may assume that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

(b) Use complex contour integration of a suitably-defined multifunction around an indented semicircle (taking account of the branch-cut also) to evaluate

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx.$$

5. (a) Show that $e^z - (4z^2 + 1) = 0$ has exactly 2 roots for $|z| \leq 1$. Hint: Use Rouché's theorem (stated below).

(b) Use the Argument Principle to show that

$$f(z) = z^7 + 1$$

has two zeros in the first quadrant.

Rouché's theorem: If f, g are analytic inside and on a simple closed contour C and $|f| > |g|$ on C, then f and $f + g$ have the same number of zeros inside C.

6. The Identity Theorem states that if a function g is analytic on a region D, and if the set of zeros of g on D has a limit point in D, then the function g must be identically zero on D.

   Let $f(z) = u(x, y) + iv(x, y)$ be a function analytic on the unit ball $B(0; 1)$.

   (a) Show that $\overline{f(\bar{z})}$ is also analytic on the unit ball.

   (b) By considering the analytic function $g(z) = f(z) + \overline{f(\bar{z})}$, show that if f takes pure imaginary values on the real axis then

   $$u(x, y) = -u(x, -y), \quad v(x, y) = v(x, -y)$$

   for all points (x, y) on the unit ball.