PhD Program in Applied Probability and Statistics Qualifying Exam B: Linear Algebra, Distributions and Statistical Inference

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1. Let θ be real and $a = \cos\theta$ and $b = \sin\theta$. Let
$$A \;=\; \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}.$$
   (a) Find the four fundamental subspaces (column space, nullspace, row space, left nullspace) of A.
   (b) Diagonalize A if possible.
   (c) Find the singular value decomposition of A if possible.

2. (a) Suppose $\lambda \neq \mu$ and let
$$A \;=\; \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{and} \qquad X_0 \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
   Define a sequence of 2×2 matrices for $\ell = 1, 2, 3, \ldots$ by
$$X_\ell \;=\; AX_{\ell-1} \;-\; X_{\ell-1}A.$$
   Find values α and β and 2×2 matrices H and K such that
$$X_\ell \;=\; \alpha^\ell H \;+\; \beta^\ell K$$
   for $\ell = 1, 2, \ldots$. Is this formula valid for $\ell = 0$? Why or why not?
   (b) Let B be a general diagonalizable 2×2 matrix, define the linear mapping L from the set of 2×2 matrices to the set of 2×2 matrices by
$$L(X) \;=\; BX \;-\; XB.$$
   Find an inner product on the vector space of 2×2 matrices for which L is selfadjoint.

3. Let A and B be $n \times n$ matrices. Suppose that the sequence $\mathbf{x}_\ell \in \mathbf{R}^n - \{\mathbf{0}\}$ for $\ell = 0, 1, 2, \ldots$ satisfies

$$A\mathbf{x}_\ell \;=\; 0 \qquad \text{and} \qquad B\mathbf{x}_{\ell+1} \;=\; \mathbf{x}_\ell.$$

Furthermore, suppose that

$$\lim_{\ell\to\infty} \mathbf{x}_\ell \;=\; \mathbf{0}.$$

Finally, suppose that the intersection of the null space (kernel) of A and the column space (range) of B is a one dimensional subspace of $\mathbf{R}^n$ having a basis containing a single vector $\mathbf{d}$.

(a) Show that $\mathbf{d}$ is an eigenvector of A. What is the eigenvalue?

(b) Show that $\mathbf{d}$ is an eigenvector of B and that the magnitude of the corresponding eigenvalue is greater than one.

4. Using a random sample $X_1, ..., X_n$ from a binomial $(k, \theta), 0 < \theta < 1, k$ is a fixed positive integer, $n \geq 2$ compute the minimum variance unbiased estimator (MVUE) of $g(\theta) \equiv P(X_1 \geq 1)$. Is the MVUE unique? Explain.

5. Let X and Y have the joint probability mass function

$$p(x, y) = \frac{e^{-2}}{x!(y-x)!}, y = 0, 1, ...; x = 0, 1, ..., y,$$

zero elsewhere. Find the moment generating function of X and Y. Also, compute the correlation coefficient of X and Y.

6. Let $X_1, ..., X_n$ and $Y_1, ..., Y_m$ be independent random samples from two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$. Find the likelihood ratio test for testing the composite hypothesis $H_o : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$. Derive an exact (not asymptotic) size α test based on the likelihood ratio test. Show work.