PhD Program in Applied Mathematics
Qualifying Exam C: Applied Mathematics

May 27, 2010

1. Consider an elastically bound, one-dimensional random walk. For a particle located at $K\triangle$ (with K an integer in the range $-R \leq K \leq R$ and $\triangle$ is a characteristic length), it moves to the right or left with probability $\frac{1}{2}\left(1 - KR^{-1}\right)$ or $\frac{1}{2}\left(1 + KR^{-1}\right)$, respectively. The duration between two consecutive steps is τ.

   (a) In the limit $\triangle \to 0$, $\tau \to 0$, $R \to \infty$, $\triangle^2/2\tau \to D$, $(R\tau)^{-1} \to \gamma$, derive the limiting equation
   $$\frac{\partial u}{\partial t} = \gamma \frac{\partial (xu)}{\partial x} + D\left(\frac{\partial^2 u}{\partial x^2}\right),$$
   for an appropriate probability density u.

   (b) Derive the steady state solution $u_e(x)$ by integrating twice and using the condition that $\int_{-\infty}^{\infty} u_e(x)dx = 1$. Interpret your solution and explain in words why u_e does not vanish to zero (as $t \to \infty$) as in the case of a free random walker.

2. A spherical jelly fish has the ability to eject or intake sea water to control its velocity and center-of-mass position while retaining its spherical shape. For this problem we assume that the jelly fish moves in a vertical straight line and experiences a gravitational force due to density mismatch, and a drag force from the surrounding viscous fluid $F_D = 6\pi\mu R u$, where μ is the fluid viscosity $R(t)$ is the jelly fish radius and u is its velocity relative to the ambient fluid. The jelly fish has a density ρ_j which may be different from the sea water density ρ_w.

   (a) Denote the jelly fish center-of-mass position as $z(t)$. The jelly fish exerts a force on the fluid by changing its radius $R(t)$. We assume that this repulsive force is proportional to the rate of change of its radius. Write down the equation of motion for the spherical jelly fish.

(b) If the jelly fish wants to stay at constant center-of-mass position, what will happen to its body size if the density $\rho_j > \rho_w$? What happens if $\rho_j < \rho_w$?

(c) Assume now that the density of the jelly fish body ρ_j varies with time such that $\rho_j - \rho_w = F(R, R_t)$. From the condition for the jelly fish to rest with z constant (as in (b)), discuss the functional form for F such that the jelly fish can have a equilibrium size while remaining at rest and stable.

3. For the differential operator and boundary conditions

$$Lu = -\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{u}{x^2}\,, \quad x \in (0,1)\,, \qquad B_1 u = u(0)\,, \quad B_2 u = u(1)\,,$$

write down the problem for the Green's function. If the Green's function exists, construct it.

(a) If the inner product of f and g has weight $w(x) = x$, so that $< f, g >= \int_0^1 f\,g\,x\,dx$, is the problem self-adjoint or not? Explain your answer, but you do not have to do any calculation.

(b) Is the associated eigenvalue problem a regular Sturm-Liouville eigenvalue problem or not? Explain.

(c) Is the operator positive-definite (i.e., $< u, Lu >> 0\ \forall\ u \in \mathcal{D}_{\mathcal{B}}$) or not, and what can you say about the eigenvalues?

4. If D is the region exterior to the unit semi-circle in the upper half-plane $D = \{(x,y)\,|y > 0\,,\ x^2 + y^2 > 1\}$, use the method of images to construct the Green's function that satisfies

$$\nabla^2 G = \delta(x-\xi)\,\delta(y-\eta)\,,$$

$$\frac{\partial G}{\partial n} = 0 \ \text{ on } \partial D\,, \qquad G = O(\ln|\boldsymbol{x}|) \ \text{ as } |\boldsymbol{x}| \to \infty\,.$$

Simplify your expression for the Green's function.

Express the solution of the boundary value problem

$$\nabla^2 u = 0\,,$$

$$\frac{\partial u}{\partial n} = \begin{cases} 0 & y = 0,\ \ |x| > 1 \\ f(\theta) & x^2 + y^2 = 1 \end{cases} \qquad |u| < \infty \text{ as } r \to \infty\,,$$

in terms of the Green's function and boundary data.

5. Show that the boundary value problem

$$\begin{aligned} \nabla^2 u - \lambda u &= h(x) \qquad x \in D\,, \\ \frac{\partial u}{\partial n} + ku &= g(x) \qquad x \in \partial D\,, \end{aligned}$$

has a unique solution provided $k > 0$ and $\lambda > 0$. Does uniqueness still hold if one of k and λ is zero while the other is strictly positive? Comment on uniqueness when both $k = 0$ and $\lambda = 0$.

6. Consider the boundary value problem

$$\nabla^2 u + k^2 u = f(x,y)\,, \quad \text{on } D = \{(x,y)|\; x \in (-\infty,\infty)\,,\; y \in (0,1)\}$$
$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0\,,$$

where $k > 0$ is given. The source term $f(x,y)$ has compact support, that is, $f(x,y)$ is identically zero outside a compact region Ω that is contained completely within D. The solution u also satisfies the boundary condition that it represents outgoing waves as $|x| \to \infty$.

(a) Construct the Green's function for this problem and use it to express the solution of the boundary value problem for u.

(b) Show that for $|x|$ sufficiently large (i.e., $|x| \gg 1$)

$$u \sim \sum_{n=0}^{M} T_n \cos(n\pi y)\, e^{ik_n x}\,,$$

where $k_n = \sqrt{k^2 - (n\pi)^2} > 0$ for $n \leq M$, and give the expression for the coefficient T_n in terms of f. Explain why the sum terminates when $n = M$.

(c) Describe the type of physical system that is modeled by this boundary value problem, and explain the significance of the finite sum in part (b) above.