PhD Program in Applied Mathematics
Qualifying Exam B: Linear Algebra and Numerical Methods

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1. Let θ be real and $a = \cos\theta$ and $b = \sin\theta$. Let
$$A \;=\; \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}.$$
   (a) Find the four fundamental subspaces (column space, nullspace, row space, left nullspace) of A.
   (b) Diagonalize A if possible.
   (c) Find the singular value decomposition of A if possible.

2. (a) Suppose $\lambda \neq \mu$ and let
$$A \;=\; \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{and} \qquad X_0 \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
   Define a sequence of 2×2 matrices for $\ell = 1, 2, 3, \ldots$ by
$$X_\ell \;=\; AX_{\ell-1} \;-\; X_{\ell-1}A.$$
   Find values α and β and 2×2 matrices H and K such that
$$X_\ell \;=\; \alpha^\ell H \;+\; \beta^\ell K$$
   for $\ell = 1, 2, \ldots$. Is this formula valid for $\ell = 0$? Why or why not?
   (b) Let B be a general diagonalizable 2×2 matrix, define the linear mapping L from the set of 2×2 matrices to the set of 2×2 matrices by
$$L(X) \;=\; BX \;-\; XB.$$
   Find an inner product on the vector space of 2×2 matrices for which L is selfadjoint.

3. Let A and B be $n \times n$ matrices. Suppose that the sequence $\mathbf{x}_\ell \in \mathbf{R}^n - \{\mathbf{0}\}$ for $\ell = 0, 1, 2, \ldots$ satisfies

$$A\mathbf{x}_\ell \;=\; 0 \qquad \text{and} \qquad B\mathbf{x}_{\ell+1} \;=\; \mathbf{x}_\ell.$$

Furthermore, suppose that

$$\lim_{\ell\to\infty} \mathbf{x}_\ell \;=\; \mathbf{0}.$$

Finally, suppose that the intersection of the null space (kernel) of A and the column space (range) of B is a one dimensional subspace of $\mathbf{R}^n$ having a basis containing a single vector $\mathbf{d}$.

(a) Show that $\mathbf{d}$ is an eigenvector of A. What is the eigenvalue?

(b) Show that $\mathbf{d}$ is an eigenvector of B and that the magnitude of the corresponding eigenvalue is greater than one.

4. The Secant method for the rootfinding problem $f(x) = 0$ is based on the linear approximation for the given function f.

(a) Derive the general iteration formula connecting x_{n+1} with x_n and x_{n-1} for the Secant method.

(b) Suppose that α is the root of f, i.e., $f(\alpha) = 0$. Prove the error formula

$$\alpha - x_{n+1} = -(\alpha - x_n)(\alpha - x_{n-1}) \cdot \frac{f''(\xi)}{2f'(\eta)}, \qquad (1)$$

with η between x_{n-1} and x_n, ξ between x_{n-1}, x_n, and α.

5. (a) Find the first three (unnormalized) orthogonal polynomials with respect to the weight function $w(x) = x$ on the interval $[0, 1]$.

(b) Calculate the coefficients w_0, w_1, x_0, and x_1 that make the approximation

$$\int_0^1 x f(x) dx = w_0 f(x_0) + w_1 f(x_1)$$

exact when f is any cubic polynomial.

6. (a) Consider solving $y' = f(x, y)$ via the scheme

$$y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + h \frac{6}{11} f(x_{n+1}, y_{n+1})$$

Determine its order, stability, and convergence property.

(b) Consider the following scheme:

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$$

Find its region of absolute stability. Is it A-stable? Explain.