Doctoral Qualifying Exam: Linear Algebra, Probability Distributions and Statistical Inference

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Problem 1.

(a) Find the set of all 2×2 singular matrices A such that

$$(x \quad y)\, A \begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 + 4xy + y^2.$$

(b) Let n be a positive integer. Find the sum of all $n \times n$ permutation matrices.

Problem 2. Let $A = U\Sigma V^T$ where

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Note that U and V are orthogonal and differ only by an exchange of the second and third column.

(a) Find an orthonormal basis for each of the four fundamental subspaces of A (the null space, the column space, the row space and the left null space).

(b) Find the largest eigenvalue and corresponding eigenvector of A.

Problem 3. Consider the initial value problem for the unkown functions $u(t)$ and $v(t)$:

$$\frac{du}{dt} + 7u - 15v = 0$$

$$\frac{dv}{dt} + 2u - 4v = 0$$

$$u(0) = u_0 \qquad v(0) = v_0$$

Find explicit formulas for $u(t)$ and $v(t)$.

Problem 4. Let $(X_1, X_2, \cdots, X_n)$ be a random sample of size n from an exponential distribution with mean $\theta > 0$.

(a) Using the *asymptotic normality* of the maximum likelihood estimator (mle) of θ, show that

$$\left(\frac{\overline{X}}{1 + \frac{a}{\sqrt{n}} z_{\alpha/2}}, \quad \frac{\overline{X}}{1 - \frac{a}{\sqrt{n}} z_{\alpha/2}} \right),$$

is a $100(1-\alpha)\%$ large sample confidence interval for the mean θ.

($a := z_{\alpha/2}$ is the upper $100(\frac{\alpha}{2})\%$ percentage point of a standard Normal r.v.)

(b) Find an *exact* likelihood ratio test of size α for testing the null hypothesis $H_0 : \theta = \theta_0$ vs. the alternatives $H_1 : \theta \neq \theta_0$.

Problem 5. Let X be a *single observation* from the following discrete distribution with probability mass function (p.m.f.)

$$f(x, \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}\,; \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

(a) Based on the single observation X, find a corresponding sufficient statistic $T(X)$ and the mle of θ.

(b) Let U be the estimator

$$U(X) = \begin{cases} 2, & \text{if } X = 1 \\ 0, & \text{otherwise} \end{cases}$$

Show U is an unbiased estimator of θ. Use Rao-Blackwell theorem to construct an improved unbiased estimator with a smaller variance.

(c) Show that the distribution of the sufficient statistic $T(X)$ is *complete*, but the distribution $f(x, \theta)$ of X is not. Does this fact imply that the improved estimator constructed in (b) is the unique UMVU estimator of θ ?

Problem 6. Let U_i; $i = 1, 2, \cdots$; be i.i.d., uniformly distributed on $(0, 1)$. Set

$$\begin{aligned} X_0 &= 1, \\ X_n &= \prod_{i=1}^{n} U_i, \quad n \geq 1 \end{aligned}$$

(a) Show that the conditional distribution of X_n given X_{n-1} is uniform on $(0, X_{n-1})$ for all $n \geq 1$.

(b) Use (a) to show, by induction, on 'n' that the moments of X_n are

$$E(X_n^k) = \left(\frac{1}{k+1}\right)^n; \quad n = 0, 1, 2, \cdots; \quad k = 0, 1, 2, \cdots,$$

by first conditioning on X_{n-1}.