Doctoral Qualifying Exam: Applied Mathematics

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1. Consider a problem of a rod of length L centimeters with insulated sides, partially immersed in the fluid of temperature $u = 0$ degrees. The upper edge of the rod ($x = 0$) is kept at $u = 0$, and the temperature at the lower end of the rod is given by Newton's law of cooling,

$$u_x(L,t) = -hu(L,t)$$

where $h > 0$ is a constant. Suppose that the initial temperature of the rod is $u(x,0) = Cx$ degrees (C is a positive constant), but instantaneously thereafter ($t > 0$) we apply our BC's. The temperature in the rod satisfies diffusion equation, with diffusion constant D.

   - Formulate nondimensional equation that governs the temperature in the rod for all $t > 0$. Specify the dimensions (units) of all constants.
   - Apply separation of variables technique to this problem. Decide whether the separation constant is positive or negative; justify your choice.
   - Write down the general solution.
   - Find an equation that determines the separation constant (you do not need to solve this equation).
   - Write down the solution that satisfies all the boundary conditions.
   - Use the initial condition to determine the remaining constants in your solution. (Note: your result for these constants may include definite integrals which you do not need to solve).
   - Write down the solution for the temperature in dimensional form, as a function of dimensional distance and dimensional time.
   - Sketch a solution for $u(x,t)$ that you would intuitively expect for few different times.

2. Plug flow reactor is an example of a chemical reactor used in industry to produce products from given reactants. Consider the following simple model for a plug flow reactor: a long tube of length L and cross-section A, where a reactant (call it B) enters at $x = 0$ at a fixed rate Q, a chemical reaction takes place as the material moves through the tube, producing the final product (call it C) as it leaves the tube at $x = L$ at the same rate Q. The flow can be assumed to be one dimensional, meaning that all the variables depend only on the axial direction, x. The rate of production of C is given by r, measured in mass/volume/time. Let $b(x,t)$ denote the concentration of B, given in mass per unit volume, and note that the flux is then given by Qb.

   - Use a basic conservation law to formulate the integral equation governing the concentration of b in some arbitrary interval $[x_1, x_2]$, $0 \leq x_1 < x_2 \leq L$.

- Assume that $r = -kb$, where k is a positive constant. Further, assume that b is sufficiently differentiable to reduce this integral conservation law to a partial differential equation for b. Write down this equation (do not attempt to solve it).

3. Consider the following boundary value problem:

$$-u'' + \alpha^2 u = f(x)\,, \tag{1}$$

where α is a real constant, and u satisfies: $u'(0) = 0$ and $u'(1) = 0$.

   (a) Find the Green's function for this problem. Does Green's function exist for all α's?

   (b) Write down the solution of (1) using the Green's function if it exists. Discuss existence and uniqueness of the solution for various α's. Give physical interpretation of your results for $\alpha = 0$ in terms of one-dimensional heat conduction.

4. Prove that the following boundary-value problem has a unique solution, provided that $k > 0$. Specifically consider the cases of $\lambda > 0$ and $\lambda = 0$.

$$\nabla^2 u - \lambda u = -h(x), \quad x \in V.$$

$$\frac{\partial u}{\partial n} + ku = f(x), \quad x \in S.$$

5. Consider the boundary-value problem in the unit square $0 < x < 1, \quad 0 < y < 1$:

$$\nabla^2 u = h(x, y).$$

$$u(0, y) = 0, \quad u(1, y) = 0,$$

$$u(x, 0) = 0, \quad u(x, 1) = f(x).$$

   (a) Construct an appropriate Green's function using a partial eigenfunction expansion in x.

   (b) Solve in terms of the Green's function.

6. Consider the BVP:

$$\nabla^2 u + k^2 u = f(x), \qquad x \in (-\infty, \infty), \quad y \in (0, 1),$$

$$u(x, 0) = 0 \quad u(x, 1) = 0,$$

$$\text{where } f(x) = \begin{cases} 0, & |x| > 1 \\ \sin(\alpha x), & |x| \le 1; \end{cases} .$$

The solution for u should also represent outgoing waves at $|x| \to \infty$. Assume that the constants k and α are real.

Construct the Green's function for this problem, and use it to solve for u.