Doctoral Qualifying Examination: Applied Mathematics
Tuesday May 27, 2008

1.) Formulate the problem of a cooling copper sphere of uniform temperature T_0, radius a, and thermal diffusion coefficient κ, thrown into icy water. Show that it takes four times as long to cool down a sphere that is twice as large in diameter.

Useful information:

(a) The Laplacian in spherical coordinates is

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

(b) To solve the ODE $r^2 y'' + 2ry' + \lambda r^2 y = 0$, where λ is a constant, use the substitution $y(r) = \frac{f(r)}{r}$ to obtain a simpler equation for $f(r)$.

2.) Consider the boundary value problem

$$u'' = f(x), \quad 0 < x < 1$$

$$\epsilon u(0) + u'(0) = 0, \quad u'(1) = 0$$

where the primes denote differentiation with respect to x.

(a) Find the Green's function for this problem.

(b) Use the Green's function to solve the boundary value problem.

(c) What happens to the solution as $\epsilon \to 0$? Does the boundary value problem still have a solution?

3.) Consider the singular eigenvalue problem

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad 0 < x < \infty$$

$$B_1 u = u(0) = 0, \quad |u| < \infty \quad \text{as} \quad x \to \infty.$$

(a) Find the eigenfunctions and eigenvalues for this problem.

(b) Solve the related Green's function problem

$$\frac{d^2G}{dx^2} + \lambda G = \delta(x - x'), \quad 0 < x < \infty$$

with the same boundary conditions as in (a).

(c) Integrate G over a large circle in the complex λ plane and determine the spectral representation of the delta function in terms of the "normalized" eigenfunctions of (a).

(d) Use your result in part (c) to deduce the Fourier sine transform and its inverse. Is there a Parseval relation between the function and its transform?

4.) Consider the boundary value problem

$$\nabla^2 u + k^2 u = f(x,y), \qquad \mathbf{x} \in (-\infty, \infty), \quad y \in (0,1)$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0$$

where $k > 0$ is given. The source term $f(x,y)$ has compact support, that is, $f(x,y)$ is identically zero outside a compact region Ω which is contained completely within D. The solution u also satisfies the boundary condition that it represent outgoing waves as $|x| \to \infty$.

(a) Construct the Green's function for this problem and use it to express the solution of the boundary value problem u.

(b) Show that for x positive and sufficiently large

$$u \sim \sum_{n=0}^{M} T_n \cos n\pi y \, e^{ik_n x}$$

where $k_n = \sqrt{k^2 - n^2\pi^2} > 0$ for $n \leq M$, and express the coefficient T_n in terms of f. Explain why the sum terminates when $n = M$, and what this corresponds to physically.

5.) Let $D = \{(x,y,z) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\}$ be a 'cubical resonator' and let ∂D denote its sides.

(a) Find the Green's function satisfying

$$G_{tt} = \nabla^2 G + \delta(\mathbf{x} - \mathbf{x}_0)\delta(t - t_0), \quad \mathbf{x} \in D$$

$$G = G_t = 0, \quad t < t_0$$

$$G = 0, \quad \mathbf{x} \in \partial D$$

where $\mathbf{x}$ and $\mathbf{x}_0$ are in D.

(b) Solve the initial-boundary value problem

$$u_{tt} = \nabla^2 u \quad \mathbf{x} \in D, \quad t > 0$$

$$u = u_t = 0, \quad t = 0$$

$$u(x, y, 0, t) = \sin(\nu t), \quad (x, y) \in \Omega$$

$$u = 0, \quad \mathbf{x} \in \partial D - \Omega$$

where Ω is a simply connected region on the face $z = 0$ of the cube. The notation $\partial D - \Omega$ describes the part of the $z = 0$ face of the cube that is outside of Ω. What happens to this solution when $\nu \to \pi\sqrt{l^2 + m^2 + n^2}$, where l, m, n are integers?

6.) Let R be the region interior to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ and ∂R its surface. Solve the boundary value problem

$$\nabla^2 u = 0, \quad (x, y, z) \in R$$

$$u = 1, \quad (x, y, z) \in \partial R$$

and prove the solution is unique.