Department of Mathematical Sciences
New Jersey Institute of Technology

Part B: Real and Complex Analysis — Doctoral Qualifying Exam, January 2013

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Assume that $f_n \to f$ uniformly on a set S. If each f_n is continuous at a point c in S, then prove that f is also continuous at c.

2. Suppose $g \in L(a, \delta)$ for every $a \in (0, \delta)$. Further assume there exists positive constants M and p such that $|g(t) - g(0+)| < Mt^p$, for every $t \in (0, \delta]$ and that $g(0+)$ exists.

   (a) Prove that the Lebesgue integral $\int_0^\delta |g(t) - g(0+)|/t \, dt$ exists.

   (b) Prove that the
$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t} \, dt = g(0+).$$

3. Consider the function
$$F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x} \, dx$$
$F(y)$ is continuous on $(0, \infty)$. Consider an increasing sequence $\{y_n\}$ such that $y_n \geq 1$, $y_n \to \infty$ as $n \to \infty$. Prove that $F(y_n) \to 0$ as $n \to \infty$ and then use the continuity of F to prove $F(y) \to 0$ as $y \to \infty$.

4. Let G be a bounded region and suppose f is continuous on $\bar{G}$ (the closure of G) and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all z on the boundary of G then either f is a constant function or f has a zero in G.

5. Let f be an entire function and let $a,\ b \in \mathbb{C}$ be distinct and such that $|a| < R$ and $|b| < R$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, evaluate $\int_\gamma [(z-a)(z-b)]^{-1} f(z) \, dz$. Use this result to give another proof of Liouville's theorem.

6. Use complex variables to calculate the following integrals, justifying all steps:
   (a) $\int_0^\infty \frac{x^{-c}}{1+x} \, dx$ for $0 < c < 1$.
   (b) $\int_0^{2\pi} \log \sin^2 2\theta \, d\theta = 4 \int_0^\pi \log \sin \theta \, d\theta$.