Department of Mathematical Sciences
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Part A: Applied Mathematics — Doctoral Qualifying Exam, January 2013

The first three questions are based on Math 613 and the next three questions are based on Math 651.

1. A simple pendulum consists of a mass m on a light inextensible string of length l, suspended from a rigid support.

   (a) Using θ to denote the angle made by the string with the vertical, and assuming air-resistance proportional to the speed of the moving mass, use Newton's 2nd law in the tangential direction on m to derive the equation of motion in the form

   $$l\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + g\sin\theta = 0. \tag{1}$$

   What does k represent here, and what are its dimensions? For a mass released from rest at some angle β what are the appropriate initial conditions on θ?

   (b) Find the two possible timescales on which one could nondimensionalize (1). Give the corresponding two dimensionless versions of (1) (note that only t needs to be rescaled since θ is dimensionless). Both dimensionless equations should contain the parameter $k/\sqrt{lg}$. Explain briefly which version is appropriate for small air resistance, and which for large air resistance.

2. The conservative PDE (in dimensionless form)

   $$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{2}$$

   with $\rho(x,t)$ representing local traffic density and $u = 1 - \rho$ representing the local (density-dependent) traffic speed, provides a simple model for the flow of traffic.

   (a) *Briefly* interpret the meaning of this equation, and of the relation between u and ρ.

   (b) Suppose the traffic is moving uniformly in the positive x-direction with density ρ_0 and speed $u_0 = 1 - \rho_0$, when a traffic signal, located at $x = 0$, suddenly turns to red. Justify the boundary conditions

   $$\rho(x,0) = \rho_0 \quad (x < 0) \quad \text{and} \quad \rho(0,t) = 1 \quad (t > 0).$$

   (c) Sketch the characteristics in $x \le 0$, $t \ge 0$, and explain why a shock forms in this region. (You do not need to sketch characteristics in $x \ge 0$.) Given that the values of ρ, u on either side of a shock at $x = S(t)$ are related to the shock speed dS/dt by the Rankine-Hugoniot conditions:

   $$\frac{dS}{dt} = \frac{[Q]}{[P]}, \quad P = \rho, \quad Q = \rho u, \quad [\cdot] \text{ denotes jump across shock,}$$

   find the shock location and hence solve the PDE everywhere in $x \le 0$, $t \ge 0$.

3. A simple dimensionless model for steady 2D flow of cold fluid, with temperature T and velocity $\boldsymbol{u} = (1, 0)$, past a heated flat plate lying along the positive x-axis, is

$$\frac{\partial T}{\partial x} = \epsilon \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$
$$T(x, 0) = 1 \quad x \geq 0$$
$$T \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty.$$

Here $\epsilon = 1/\mathrm{Pe} \ll 1$ is an inverse Peclet number.

   (a) Demonstrate that a regular asymptotic expansion $T = T_0 + \epsilon T_1 + \cdots$ cannot yield a solution that satisfies all boundary conditions.
   (b) Look for a solution that includes a thermal boundary layer near the plate by rescaling: $y = \delta Y$, where δ depends on ϵ (you should find the appropriate value of δ in terms of ϵ).
   (c) Writing the solution within this thermal boundary layer as $\hat{T}(x, Y)$ and seeking an appropriate asymptotic expansion for $\hat{T}$, show that the leading order solution $\hat{T}_0$ satisfies

$$\frac{\partial \hat{T}_0}{\partial x} = \frac{\partial^2 \hat{T}_0}{\partial Y^2}, \quad \hat{T}_0(x, 0) = 1 \ (x \geq 0), \quad \hat{T}_0 \to 0 \text{ as } Y \to \infty.$$

   By seeking a similarity solution $\hat{T}_0 = F(Y/x^{1/2})$, solve for $\hat{T}_0$.
   You may assume that

$$\int_0^\infty e^{-s^2/4} ds = \sqrt{\pi}.$$

4. (a) Find the general solution of $2t^2 y'' + 3ty' - y = 2t^{\frac{3}{2}}$.
   (b) Find the first 4 nonzero terms in each of the two linearly independent solutions of $2x^2 y'' - xy' + (1 + x)y = 0$ about $x = 0$.

5. Consider the initial boundary value problem

$$u_t = u_{xx} - \alpha u \ \ ; \ x \in (0, 1), \ t > 0$$
$$u(x, 0) = x \ \ ; \ x \in (0, 1)$$
$$u_x(0, t) = 0 \ \ ; \ t \geq 0$$
$$u_x(1, t) - u(1, t) = 0 \ \ ; \ t \geq 0$$

where $\alpha > 0$ is a constant. This problem models a rod whose lateral surface area is not insulated (heat flows out of the sides) but whose end-point at $x = 0$ is insulated while its end-point at $x = 1$ allows heat to enter at a rate proportional to the temperature there.

   (a) Find the value of α for which the problem above has equilibrium, time-independent (steady), solutions $u(x, t) = U(x)$, where $U(x)$ satisfies an ordinary differential equation boundary value problem.
   (b) Solve the initial boundary value problem with the method of separation of variables and find the value of α that assures $u(x, t) \to 0$ as $t \to \infty$. You will need to carefully analyze the eigenvalue problem that results from the method of separation of variables.

6. Completely solve the non-homogeneous heat equation on an infinitely-long bar,

$$u_t = Du_{xx} + \frac{2t}{x^2+t^2} \quad ; \quad |x| < \infty,\ t > 0,$$

with initial condition

$$u(x,0) = 1\ ;\ |x| < a, \quad u(x,0) = 0\ ;\ |x| > a.$$

NOTE: For full credit your answer must only involve integrals over x and t. Make use of the supplied table of Fourier transforms.