Preliminary Exam in Real and Complex Analysis: January 2012

1. Use Fourier series analysis to prove that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} > 0, \quad 0 < x < \pi$$

by finding a suitable function on $[0, 2\pi]$ whose Fourier series is the given one. Be sure to cite all necessary theorems to justify your proof.

2. Let f be a non-negative measurable function on a measure space (X, M, μ) and define $\lambda(E) = \int_E f \, d\mu$ for $E \in M$.

   (a) Prove that λ is a measure on M.

   (b) Prove that if g is a non-negative measurable function on (X, M, μ) that $\int g \, d\lambda = \int fg \, d\mu$. *Hint*: Prove first for g as a simple function, then use the monotone convergence theorem.

3. Assume that $f_n(x) \to f(x)$ uniformly on a set S. If each f_n is continuous at a point c, prove that f is also continuous at c. Give an example showing that if the assumption of uniform convergence is replaced with pointwise convergence that the conclusion of the continuity of $f(x)$ at c may fail.

4. **Complex integration** Find the improper integral below by analytically extending the integrand into the region bounded by a semi-circular contour indented at the origin, choosing an appropriate branch of the logarithm. Make sure to carefully justify all steps:

$$\int_0^{\infty} \frac{\ln x \, dx}{x^2 + a^2}$$

5. **Cauchy Residue Theorem and Inversion Mapping** Consider the following integral over a circular contour of radius $1/4$:

$$\int_{|z|=1/4} \frac{\exp\left(\frac{i}{z}\right)}{z \sinh\left(\frac{1}{z}\right)} dz$$

   (a) Categorize all singularities of the integrand inside the integration contour. Explain why the Cauchy Residue Theorem cannot be directly applied to this integral.

   (b) Calculate the value of the integral using the mapping (variable transformation) $w = 1/z$. *Hint*: be careful with the mapping of the integration contour.

(Over please)

6. **The Big Picard Theorem and the Liouville Theorem** The Liouville Theorem states that the modulus of a non-constant entire function cannot be bounded.

   The "Big" Picard Theorem states that if an analytic function $f(z)$ has an essential singularity at a point z_0, then on any open set containing z_0, $f(z)$ attains *all* possible complex values, with at most a single exception, infinitely many times.

   (a) Prove that any *entire* function $f(z)$ is either a polynomial or has an essential singularity at $z = \infty$. *Hint*: consider its series expansion about $z = 0$.

   (b) Use the above result to prove that Liouville's Theorem directly follows from the Big Picard Theorem.

   (c) Verify by direct calculations that the Big Picard Theorem is satisfied in the special case of function $f(z) = e^{1/z}$ near $z_0 = 0$. *Hint*: consider solutions of equation $f(z) = w$. In the case of this function, the "single exception" mentioned in the theorem is the value zero.