1. This problem has two independent parts.

(i) Suppose that X_1, X_2, X_3 are independent and identically distributed with common probability density function given by $f(x) = e^{-x}, x > 0$, zero elsewhere. Find $P(X_1 < X_2 < X_3 | X_3 < 1)$.

(ii) Suppose that $X_1, \dots, X_n$ are independent and identically distributed exponential random variables with parameter λ. Find the distribution of $T = 2\lambda \sum_{i=1}^{n} X_i$ and identify it.

2. A random variable (r.v.) X is said to have a *two-parameter inverse Gaussian* distribution with parameters $\mu > 0$ and $\lambda > 0$ (and denoted by $X \sim$ IG(μ, λ)), if it has a probability density function (p.d.f.)

$$g(x|\mu, \lambda) := \left(\frac{\lambda}{2\pi}\right)^{1/2} x^{-3/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, x > 0.$$

(i) Use the identity

$$\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty x^{-3/2} \exp\left\{-\frac{\lambda x}{2\mu^2} - \frac{\lambda}{2x}\right\} dx = \lambda^{-1/2} \exp\left(-\frac{\lambda}{\mu}\right)$$

to show that the joint moment generating function of X and $W = X^{-1}$ is given by

$$M_{X,W}(t, u|\lambda, \mu) = \left(\frac{\lambda}{\lambda - 2u}\right)^{1/2} \exp\left(\frac{\lambda}{\mu} - \frac{\sqrt{(\lambda - 2u)(\lambda - 2\mu^2 t)}}{\mu}\right).$$

(ii) Deduce the moment generating function of X from part (i) when $t < \lambda/2\mu^2$.

(iii) Use part **2(ii)** to show that if $X_1, \dots, X_n$ are independent IG(μ_i, λ_i) variables $i = 1, \dots, n$, then the distribution of $\sum_{i=1}^{n}(\mu_i^{-2}\lambda_i X_i)$ is IG(μ, λ) with $\mu = \sum_{i=1}^{n}\left(\frac{\lambda_i}{\mu_i}\right), \lambda = \mu^2$.

3. This problem has two independent parts.

(i) Let the probability mass function $p(x)$ be positive on and only on the nonnegative integers. Given $p(x) = 4p(x-1)/x, x = 1, 2, 3, \dots$, find $p(x)$ and identify its name.

(ii) Let $F(x)$ be the cumulative distribution function of a continuous random variable X. Assume that $F(0) = 0$ and $0 < F(x) < 1$ for $x > 0$. Suppose that X satisfies

$$P(X > x + y | X > x) = P(X > y).$$

Show that $F(x) = 1 - e^{-\lambda x}$, where $\lambda > 0$.

4. Observations $Y_1, \ldots, Y_n$ are described by the relationship $Y_i = \beta_0 + x_i\beta_1 + \epsilon_i$, where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are independent with common distribution $\mathrm{N}(0, \sigma^2)$.

(i) Derive the $(1-\alpha)$ confidence interval for β_0.

(ii) Use the derivation in **4(i)** to also derive a test for testing $H_0 : \beta_0 = 0$ versus $H_1 : \beta_0 \neq 0$. Clearly state all expressions.

5. Consider the simple linear regression model, as described in problem **4.**

(i) Define the vector of residuals $\mathbf{e}$ in terms of the hat matrix.

(ii) Derive the variance of $\mathbf{e}$ in terms of the hat matrix.

(iii) Use **5(ii)** to obtain the estimate of the variance of $\mathbf{e}$.

6. An analyst decided to fit the multiple regression model
$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1}X_{i2} + \beta_5 X_{i1}X_{i3} + \beta_6 X_{i2}X_{i3} + \epsilon_i$, where $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$, $i = 1, \ldots, 20$. To reduce correlation between the covariates in this model, the centered variables $x_{i1} = X_{i1} - \hat{X}_1 = X_{i1} - 25.305$, $x_{i2} = X_{i2} - \hat{X}_2 = X_{i2} - 51.170$, and $x_{i3} = X_{i3} - \hat{X}_3 = X_{i3} - 27.620$ is used. The fitted regression equation is given by
$\hat{Y} = 20.53 + 3.43x_1 - 2.095x_2 - 1.616x_3 + 0.00888x_1x_2 - 0.08479x_1x_3 + 0.09042x_2x_3$,
$MSE = 6.745$, where the true model is
$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i1}x_{i2} + \beta_5 x_{i1}x_{i3} + \beta_6 x_{i2}x_{i3} + \epsilon_i$.

One would like to test whether the interaction terms between the three predictor variables should be included in the regression model. Use the above information and the Table 1. below to conduct a partial F-test at 5% significance level. Clearly state the null and alternate hypotheses, test statistic, decision rule and the conclusion.

Table 1.

Variable	**Extra Sum of Squares**	Value
x_1	$SSR(x_1)$	$= 352.270$
x_2	$SSR(x_2 \mid x_1)$	$= 33.169$
x_3	$SSR(x_3 \mid x_1, x_2)$	$= 11.546$
x_1x_2	$SSR(x_1x_2 \mid x_1, x_2, x_3)$	$= 1.496$
x_1x_3	$SSR(x_1x_3 \mid x_1, x_2, x_3, x_1x_2)$	$= 2.704$
x_2x_3	$SSR(x_2x_3 \mid x_1, x_2, x_3, x_1x_2, x_1x_3)$	$= 6.515$
F(0.975, 3, 13) = 4.3472,	F(0.95, 3, 13) = 3.4105, F(0.975, 7, 19) = 3.0509,	
F(0.95, 7, 19) = 2.5435,	F(0.95, 4, 13) = 3.1791, F(0.975, 4, 13) = 3.9959,	
F(0.95, 4, 19) = 2.8951,	F(0.975, 4, 19) = 3.5587, F(0.975, 3, 19) = 3.9034,	
F(0.95, 3, 19) = 3.1274.		