Preliminary Exam in Applied Mathematics: January 2012

1. Given two species, interdependence might occur because one species (the "prey") serves as a food source for the other species (the "predator"). The growth rate of the prey population is determined by the equation:

$$\frac{dN(t)}{dt} = KN(t)\left(1 - \frac{N(t)}{K}\right) - SM(t), \quad t > 0,$$

where K and S are parameters, and the growth rate of the predator population is determined by the equation:

$$\frac{dM(t)}{dt} = -RM(t) + QN(t), \quad t > 0,$$

where R and Q are parameters. Consider $N(0) = N_0$ and $M(0) = M_0$, where $N_0 > 0$ and $M_0 > 0$.

a) Briefly explain what each term in the system of equations represents.
b) If there are no predators, so that there is only a "prey" population, find reasonable scales and nondimensionalize the problem. Find the equilibrium solutions of the dimensionless problem, and use a linear stability analysis to determine whether the equilibrium solutions are stable or unstable.

2. **a)** Formulate a model for the conservation of cars in a flow of traffic. Let ρ be the traffic density and let q be the flux of traffic (i.e. number of cars per hour passing a fixed location). Recall that the traffic flux q equals the density of cars ρ times their velocity u.
b) Carry out a linear analysis for small disturbances $\phi(x,t)$ about the equilibrium solution $\rho = \rho_0 =$ constant. Find the solution for $\phi(x,t)$ and describe it. How does the solution behave?
c) Suppose that $q = \frac{1}{2}\rho^2$ (i.e. $u = \frac{1}{2}\rho$) and the initial traffic is:

$$\rho(x,0) = \begin{cases} a & x < 1\,, \\ b & x > 1\,. \end{cases}$$

Consider the two cases $a > b$ and $a < b$, and determine the traffic density (for $t > 0$) in each case. Explain your answer for each case. Sketch the characteristics in the $x - t$ plane for each case.

3. Consider the one-dimensional flow of a fluid through a nozzle of a varying cross-section $A(x)$.
   a) Derive from first principles the equation for conservation of mass:

$$A(x)\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u A(x))}{\partial x} = 0.$$

   b) Conservation of momentum for this system is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

   Derive the linearized wave equation for sound propagation in the nozzle, assuming an adiabatic equation of state relating (small) pressure perturbations p' to the density perturbations ρ', i.e. $p' = c^2\rho'$, where c is the speed of sound.

4. (a) Given one solution of the homogeneous equation is $y = x$, determine the general solution of the nonhomogeneous equation

$$(1 - x^2)y'' - 2xy' + 2y \;=\; 6.$$

   (b) Find the general solution to the differential equation

$$y^{(4)} - 4y^{(3)} + 7y'' - 6y' + 2y \;=\; \sin(x).$$

   It is sufficient to write down the particular solution using the method of undetermined coefficients without calculating the coefficients.

5. Find the eigenvalues and eigenfunctions to the eigenvalue problem

$$y'' + \frac{\lambda y}{(1+x)^2} \;=\; 0, \quad \text{for } x \in (0, b), \quad y(0) = 0, \quad y(b) = 0.$$

   Hint: Make a change of variables $e^{\xi} = x + 1$.

6. Solve Laplace's equation inside a rectangle ($0 \le x \le L$, $0 \le y \le H$) with the solution prescribed at the boundary as $u(0, y) = g(y)$, $u(L, y) = 0$, $u(x, 0) = 0$ and $u(x, H) = f(x)$. Based on your solution, briefly discuss how you would solve the heat equation inside the same rectangle with the same boundary conditions, given the initial condition $u(x, y, t = 0) = u_0(x, y)$ for $0 \le x \le L$ and $0 \le y \le H$.