Qualifying Exam in Applied Mathematics: January 2011

1. Consider the initial value problem

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = 0,$$

$$\phi(x,0) = \begin{cases} \phi_L, & x < 0 \\ \phi_R, & x > 0; \end{cases} .$$

   (a) Under what conditions will a solution to the above contain a shock?

   (b) Under the conditions identified above, use characteristics to construct a solution. Also, derive an equation describing the evolution of the shock. Sketch in the $x-t$ plane.

2. Find the Green's function $G(x,\xi)$ for the two-point boundary value problem

$$\frac{d^2u}{dx^2} = f(x), \qquad x \in (0,1),$$

$$\frac{du}{dx}(0) = c_1, \quad \frac{du}{dx}(1) + \epsilon u(1) = c_2.$$

   Write the solution for $u(x)$ in terms of $G(x,\xi)$.

   What happens to the solution u as $\epsilon \to 0$, and under what condition on the data (f, c_1, c_2) does the solution remain bounded?

3. Find the solution of

$$\frac{d^2u}{dx^2} + \mu u = -f(x), \qquad x \in (0,1),$$

$$u(0) = c_1, \quad \frac{du}{dx}(1) = c_2,$$

   where μ is a parameter, as an expansion in terms of eigenfunctions of the associated homogeneous problem $u'' + \lambda u = 0$ with eigenvalue λ. Consider the case where μ is not an eigenvalue, and use the solution to identify the eigenfunction expansion of the Green's function.

   Write the solution in the form $u = u_f + u_c$, where u_f is the response to the forcing in the ode alone (i.e., with $f(x) \neq 0$ and $c_1 = c_2 = 0$) and u_c is the response to the boundary data alone (i.e., with $c_1 \neq 0$, $c_2 \neq 0$ and $f(x) = 0$). What type of convergence do you expect for the series for u_f and u_c. Explain your answer.

4. Find the Green's function for the problem

$$u_t = u_{xx} \quad x > 0, \; t > 0,$$

$$u(x,0) = f(x), \quad u(0,t) = g(t),$$

where $u(x,t)$ is bounded as $x \to \infty$.

When $f(x) = 1$ and $g(t) = 0$, name three different methods for constructing the solution $u(x,t)$. Develop one of these methods to construct the solution, reducing your answer to its simplest form.

You can quote the following results: The free-space Green's function of the diffusion equation is $f(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$, which has Laplace transform $F(x,s) = \frac{e^{-x\sqrt{s}}}{2\sqrt{s}}$. If $f(t)$ has Laplace transform $F(s)$, then $\int_0^t f(\tau)d\tau$ has Laplace transform $F(s)/s$. The error function is $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$.

5. The Dirichlet problem for Laplace's equation in the upper half-space in 3D is

$$\begin{aligned} \nabla^2 u = 0, \quad & \boldsymbol{x} = (x,y,z) \quad z > 0, \\ u = g(\boldsymbol{x}) \quad & \text{on } z = 0, \ (x,y) \in \Omega, \end{aligned}$$

where Ω is a compact subset of $z = 0$ and $u = 0$ elsewhere on $z = 0$. State the boundary value problem satisfied by the Green's function $G(\boldsymbol{x}, \boldsymbol{\xi})$.

(a) Use Green's theorem to derive the Poisson formula for $u(\boldsymbol{x})$ in terms of the Green's function, and construct the Green's function by the method of images. Show that the solution for $u(\boldsymbol{x})$ can be expressed as

$$u(\boldsymbol{x}) = \frac{z}{2\pi} \int_{\boldsymbol{\xi} \in \Omega} \int \frac{g(\boldsymbol{\xi})}{|\boldsymbol{x} - \boldsymbol{\xi}|^3} \, dS_{\boldsymbol{\xi}} \, .$$

(b) Find the first term in the far-field expansion for $u(\boldsymbol{x})$ as $|\boldsymbol{x}| \to \infty$. What type of leading order field do you find, monopole, dipole etc?

6. Consider the initial boundary value problem

$$\begin{gathered} u_{tt} - u_{xx} = 0 \quad x > 0, \ t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = h(x), \\ u(0,t) = g(t) \, . \end{gathered}$$

(a) With zero initial data $f = h = 0$ but $g \neq 0$, construct the solution for $u(x,t)$. You should find that $u(x,t) = H(t-x)g(t-x)$ where H is the unit step function.

(b) Find the solution for arbitray data $f \neq 0$, $h \neq 0$, $g \neq 0$.

(c) From your answer to (b), is there initial data for f and h in terms of a boundary signal $g \neq 0$ so that in the region $0 < x < t$: (i) $u \equiv 0$, or (ii) there is no right-going wavefield. Explain your answer.