Preliminary Exam in Applied Mathematics: January 2011

1. Consider the problem of 1-D heat conduction through a solid metal pipe of varying cross-section, $A(x)$. Use the general balance law to derive the partial differential equation describing this system. You may assume the density, ρ, specific heat, c_p, and thermal conductivity, λ, are constant. (Recall: the quantity $\rho c_p T$ represents the heat energy per unit volume).

2. Consider the initial value problem

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = 0,$$

$$\phi(x,0) = \left\{ \begin{array}{ll} \phi_L, & x < 0 \\ \phi_R, & x > 0; \end{array} \right. .$$

   (a) Under what conditions will a solution to the above contain a shock?

   (b) Under the conditions identified above, use characteristics to construct a solution. Also, derive an equation describing the evolution of the shock. Sketch in the $x-t$ plane.

3. The 1D equations governing a shallow layer of water with rotation are:

$$h_t + (uh)_x = 0,$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = \gamma hv,$$

$$(hv)_t + (huv)_x = -\gamma hu,$$

where $h(x,t)$ is the height of the water, $u(x,t)$ and $v(x,t)$ are the velocity components, and the constant, γ is the Coriolis parameter. Consider a small perturbation from a uniform state, i.e.

$$h = 1 + \epsilon H(x,t), \quad u = \epsilon U(x,t), \quad v = \epsilon V(x,t).$$

Determine the linearized equations for $H(x,t)$, $U(x,t)$, and $V(x,t)$, and then derive from them a single second order PDE for $U(x,t)$. Construct a (right) traveling wave solution to the resulting equation of the form $U(x,t) = U(x-ct)$, with $c > 1$, on the interval $-\infty < x < \infty$, with the initial condition below, and determine the wave speed, c:

$$U(x,0) = \sin x.$$

4. Find the general solution to the following second order differential equations.

   (a)

$$y'' - 2y' + y = e^t$$

   (b)

$$x^2y'' - 4xy' + 6y = x^4 \sin x$$

5. (a) Find the general solution to the differential equation

$$t^2y' - 2ty = t^4 \cos(t).$$

   (b) Find the general solution to the differential equation and explain what value does your solution approach as $t \to \infty$?

$$t\frac{dY}{dt} = Y \ln Y \left(1 - \ln Y\right).$$

6. Consider Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = g(y), \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

   (a) Show that $u(x, y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants.

   (b) Find the general solution for non-constant $f(x)$ and $g(y)$).