Qualifying Exam in Linear Algebra and Numerical Methods: January 2011

1. Suppose
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Determine the dimension of each of the four fundamental subspaces of $A$.

   (b) Give a basis for the null space of $A$.

2. (a) For a real symmetric matrix show that the eigenvalues are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

   (b) Find two subspaces, $U$ and $V$, of $\mathbf{R}^{\mathbf{2\times 2}}$ such that $W = U \cup V$ is the set of all matrices in $\mathbf{R}^{\mathbf{2\times 2}}$ having an orthogonal eigenbasis.

3. Let $a > 0$. Newton's method for finding the roots of $f(x) = x^2 - a$ is the iteration
$$x_{\ell+1} \;=\; \frac{1}{2}\left(x_\ell \;+\; \frac{a}{x_\ell}\right).$$

   (a) Suppose that $x_0 = 1$. Show that Newton's method produces a sequence $\{x_\ell\}$ such that $x_\ell \to \xi$ as $\ell \to \infty$ where $\xi^2 = a$.

   (b) Let $n$ be a positive integer and suppose $A \in \mathbf{R}^{\mathbf{n\times n}}$ is positive definite. Let $I$ be the $n \times n$ identity matrix. Show that the iteration
$$X_{\ell+1} \;=\; \frac{1}{2}\left(X_\ell \;+\; AX_\ell^{-1}\right)$$
   with $X_0 = I$ converges to an $n \times n$ matrix $\Xi$ such that $\Xi^2 = A$.

4. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$. Find the minimax approximation to $f(x)$ on $[-1, 1]$ by a polynomial of degree $\leq n - 1$, and also find the minimax error $\rho_{n-1}(f)$.

5. Consider a quadrature rule for computing the integral $\int_0^1 f(x)dx$ using two quadrature nodes with 0 as one node.
   (a) What is the highest degree of precision of such quadrature?
   (b) Find the weights and the other node which achieves the highest degree of precision.

6. The modified Euler's method
$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n))$$
   is used to obtain a numerical solution to $y' = f(t, y)$.

   (a) Show that this numerical scheme is second order accurate.

   (b) What is the region of absolute stability of this scheme? Is the scheme A-stable? Explain.