Doctoral Qualifying Exam: Real and Complex Analysis

January 11, 2010

Problem 1

Consider the sequence $\{f_n\}$ of even, Lebesgue-measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ defined for each positive integer n and $x \geq 0$ as

$$f_n(x) := \begin{cases} 1, & x \in \bigcup_{k=1}^{\infty} E_{k,n} \\ 0, & x \notin \bigcup_{k=1}^{\infty} E_{k,n} \end{cases},$$

where $E_{k,n} := \{x \in \mathbb{R} : k - \frac{1}{2n} - \frac{1}{4n^k} \leq x \leq k - \frac{1}{2n}\}$. Determine whether or not each of the following is true, explaining your answer in each case: (i) $\{f_n\}$ converges uniformly on $\mathbb{R}$; (ii) $\{f_n\}$ converges in the $L^1(\mathbb{R})$ norm; (iii) $\{f_n\}$ converges almost uniformly on $\mathbb{R}$; (iv) $\{f_n\}$ converges in (Lebesgue) measure on $\mathbb{R}$; and (v) $\{f_n\}$ converges almost everywhere on $\mathbb{R}$.

Problem 2

Consider the integral equation

$$\varphi(x) = 1 + \int_0^{\infty} \exp\left[-(x+2)\, t\varphi(t)\right] dt, \tag{1}$$

where $\varphi \in X := BC([0,\infty)) := \{\varphi : [0,\infty) \to \mathbb{R} : \varphi \text{ is bounded and continuous}\}$ and $\mathbb{R}$ denotes the real numbers. Verify the following:

(a) A solution of (1) must satisfy $1 \leq \varphi(x) \leq 3/2$ for all $x \in [0,\infty)$.

(b) $\varphi(x) \to 1$ as $x \to \infty$ for any solution of (1).

(c) Equation (1) has a unique solution in X.

Also show how one can approximate the solution, and find at least one such approximation.

Problem 3

Prove the Riemann-Lebesgue type result that if $f \in L^1(\mathbb{R})$, then

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(t) e^{-\lambda t^2}\, dt = 0.$$

Problem 4

(a) By integrating a suitable complex function around a large sector of angle $\pi/4$ in the complex z-plane, show that

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \sqrt{\pi/8}.$$

(These integrals exist only as improper integrals.)
You may assume that

$$\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$$

(b) Use complex contour integration of a suitably-defined multifunction to prove that

$$\int_0^\infty \frac{x^{a-1}\,dx}{1+x} = \pi \mathrm{cosec} \pi a, \qquad 0 < a < 1.$$

Problem 5

Let the function $f(z)$, with Taylor series $f(z) = \sum_{n=0}^\infty c_n z^n$, be analytic on $B(0; R)$, $R > 0$.

(a) Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta = \sum_{n=0}^\infty |c_n|^2 r^{2n}, \qquad 0 \le r < R.$$

HINT: You will need to justify interchanges of summation and integration.

(b) Suppose that f is analytic and bounded on $\mathbb{C}$. Use (a) to deduce that f is constant.

Problem 6

The Identity Theorem states that if a function f is analytic on a region D, and if the set of zeros of f on D has a limit point in D, then the function f must be identically zero on D.

Let $\{z_n\}$ be a sequence of distinct points in $B(0; 1)$ such that $z_n \to 0$ as $n \to \infty$. Decide whether statements (a)–(c) are true or false, for all choices of $\{z_n\}$:

(a) If f is analytic on $B(0; 1)$ and $f(z_n) = \sin z_n$ for all n, then $f(z) = \sin z$ for all $z \in B(0; 1)$.
(b) There exists f analytic on $B(0; 1)$ such that $f(z_n) = n$ for all n.
(c) There exists f analytic on $B(0; 1)$ such that

$$f(z_n) = \begin{cases} 0 & n \text{ even} \\ z_n & n \text{ odd.} \end{cases}$$

Demonstrate the importance of the theorem's requirement that the limit point of zeros lie *in* the function's domain of analyticity by considering possible functions f such that $f(z_n) = \sin(1/z_n)$ for all n (for some sequence with $z_n \to 0$ as $n \to \infty$) and showing that $f(z) - \sin(1/z)$ does not have to be identically zero in this case.