Doctoral Qualifying Exam: Linear Algebra, Probability Distributions, and Statistical Inference

January 11, 2010

Problem 1.

(a) Determine the values of α for which the following two matrices are similar:

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Prove your result.

(b) Let $\epsilon > 0$ and diagonalize

$$\begin{pmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{pmatrix}.$$

What happens to this diagonalization when $\epsilon \to 0$?

Problem 2. Let $\alpha \in \mathbf{R}$ and

$$A \;=\; \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

(a) Find

$$P \;=\; \lim_{t\to\infty} e^{At}.$$

(b) Compute $P^2 - P$.

(c) What is the column space of P?

(d) Find a positive definite matrix B such that $BP = P^t B$.

Problem 3. Let A be a real symmetric 2×2 matrix that is not a scalar multiple of the identify matrix. Consider the linear mapping L on the vector space of real 2×2 matrices defined by

$$L(X) \;=\; AX - XA.$$

Find the eigenvalues and corresponding eigenvectors of the linear mapping L.

Problem 4. This question has two independent parts.

(a) Suppose that X and Y are iid $N(0,1)$ random variables. Define $Z = \min(X, Y)$. Prove that Z^2 has a Chi-squared distribution and find its degrees of freedom.

(b) Let Y_1 and $Y_2\,(> Y_1)$ be the *order statistics* from a random sample of size 2 from a Normal distribution $N(0, \sigma^2)$. Prove that $E(Y_1 + Y_2) = 0$, without evaluating either expectation, and then show,

$$EY_1 = -\frac{\sigma}{\sqrt{\pi}}$$

Problem 5. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta}, -\theta < x < \theta, \theta > 0.$$

(a) Obtain a complete sufficient statistic.

(b) Obtain a best unbiased estimator of θ.

Problem 6. Let X be one observation from a Cauchy(θ) distribution. That is, $f(x|\theta) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, -\infty < x < \infty, -\infty < \theta < \infty$.

(a) Show that this family does not have a monotone likelihood ratio.

(b) Show that the test $\phi(x) = 1$, if $1 < x < 3$ and $\phi(x) = 0$ otherwise is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

(c) Calculate the Type I and Type II Error probabilities.