Doctoral Qualifying Exam: Applied Mathematics

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1. Consider the dimensionless reaction diffusion equation

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

(a) Assume that there exists a traveling wave solution $u(x,t) = U(z)$ with $z = x - ct$ where c is the wavespeed. Write down the equation for U.

(b) Further assume that for $0 < \epsilon \equiv \frac{1}{c^2} \le 0.25$, a non-negative monotone solution for $U(z) = g(\xi = \epsilon^{1/2} z)$ exists with $U(-\infty) = 1$ and $U(\infty) = 0$. Show that, in terms of $g(\xi)$, the equation for U from (a) can be recast as

$$\epsilon \frac{d^2 g}{d\xi^2} + \frac{dg}{d\xi} + g(1-g) = 0,$$

with $g(-\infty) = 1$, $g(\infty) = 0$.

(c) Setting $g(0) = 1/2$, we look for solutions of (b) as a regular perturbation series in ϵ:

$$g(\xi) = g_0(\xi) + \epsilon g_1(\xi) + \epsilon^2 g_2(\xi).$$

Show that

$$\frac{dg_0}{d\xi} = -g_0(1-g_0),$$

with $g_0(-\infty) = 1$, $g_0(\infty) = 0$, and $g_0(0) = \frac{1}{2}$. Find the solution for $g_0(\xi)$.

(d) Show that the equation for g_1 is

$$\frac{dg_1}{d\xi} + (1 - 2g_0)g_1 = -\frac{d^2 g_0}{d\xi^2},$$

with $g_1(\pm\infty) = 0$ and $g_1(0) = 0$. Find the solution g_1 and write down the traveling wave solution in terms of the orginal variable U and z.

2. Consider the competition model for two species with populations N_1 and N_2

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right), \tag{2}$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2}\right). \tag{3}$$

Non-dimensionalize the system and recast the equations in terms of the new variables $u_1 = N_1/K_1$, $u_2 = N_2/K_2$, and scaled time $\tau = r_1 t$. Invistigate the stability of this system and sketch the phase plane trajectories. Briefly describe under what conditions the species N_2 becomes extinct.

3. A thin insulated semiconductor rod occupying a region $0 < x < L$ is subject to illumination with intensity $I = I_0(1 - \cos(\pi x/L))$ photons/(meter $\cdot$ second). As a result, charge carriers are created with the rate $\alpha I(x)dx$ particles/second in an interval $[x, x+dx]$, and the resulting particles diffuse with diffusion constant D meters^2/second and are annihilated with the rate n/τ_r particles /(meter $\cdot$ second), where n is the particle line density (in particles/meter) and τ_r is the recombination time (in seconds).

(a) Assuming initially no charge carriers were present, write down the initial boundary value problem satisfied by the charge carrier density $n = n(x,t)$ (*Hint:* write down the conservation law for the charge carrier density n, taking into consideration the particle diffusion with flux $j = -D\,\partial n/\partial x$, as well as sources and sinks).

(b) Show that, up to a rescaling, this initial boundary value problem is equivalent to

$$u_t = u_{xx} - u + 1 - \cos(\pi x/l),\ 0 < x < l, \qquad u_x|_{x=0} = u_x|_{x=l} = 0, \qquad u|_{t=0} = 0, \tag{4}$$

for some l.

(c) Solve the initial boundary value problem in Eq. (4).

(d) What is the asymptotic behavior of the obtained solution as $t \to \infty$? What boundary value problem does the limit solution satisfy?

4. Consider the operator L and the boundary conditions B_1 and B_2 defined for all $u \in C^2(0,l) \cap C^1([0,l])$ as follows:

$$Lu = -\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{9u}{4x^2}, \qquad B_1(u) = u(0), \qquad B_2(u) = \frac{du(l)}{dx}, \tag{5}$$

(a) Show that all the eigenvalues of L with boundary conditions $B_1 = 0$ and $B_2 = 0$ are strictly positive.

(b) Does the Green's function of this boundary value problem exist? If so, compute it.

(c) State all the main properties of the eigenvalues of L with the considered boundary conditions (justify your answer). Do the eigenfunctions of L form a complete orthonormal set? In which space?

5. Consider the boundary value problem for $u = u(x,y,z)$:

$$u_{xx} + u_{yy} + u_{zz} + f(x,y,z) = 0 \text{ in } y > 0, \qquad u_y = 0 \text{ on } y = 0. \tag{6}$$

(a) Use the method of images to find the Green's function G for this problem which vanishes at infinity.

(b) Assuming the convolution of f with G exists, write down the formula for the generalized solution u.

(c) Suppose

$$f(x,y,z) = \begin{cases} \frac{3q}{\pi R^4}(R - \sqrt{x^2+y^2+z^2}), & 0 \le x^2+y^2+z^2 < R^2, \\ 0, & x^2+y^2+z^2 \ge R^2. \end{cases} \tag{7}$$

Find the solution u and show that it is equal to the solution generated by a point source $f_q = q\delta(x)\delta(y)\delta(z)$ for $\sqrt{x^2+y^2+z^2} > R$. Is the obtained solution classical?

6. (a) Solve the generalized initial value problem for $u = u(x,t)$, with $-\infty < x < \infty$ and $t \ge 0$:

$$u_{tt} = u_{xx}, \qquad u(x,0) = \operatorname{sech} x, \qquad u_t(x,0) = -\tanh x \operatorname{sech} x. \tag{8}$$

Is the solution classical?

(b) Now solve the same problem with $x > 0$ instead, supplemented with the boundary condition $u_x(0,t) = 0$. Is the solution classical?

(c) Describe the qualitative behavior of the solutions obtained in (a) and (b) above (sketch the solution for a few values of t).