Doctoral Qualifying Exam: Linear Algebra and Numerical Methods

January 11, 2010

Problem 1.

(a) Determine the values of α for which the following two matrices are similar:

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Prove your result.

(b) Let $\epsilon > 0$ and diagonalize

$$\begin{pmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{pmatrix}.$$

What happens to this diagonalization when $\epsilon \to 0$?

Problem 2. Let $\alpha \in \mathbf{R}$ and

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

(a) Find

$$P = \lim_{t \to \infty} e^{At}.$$

(b) Compute $P^2 - P$.

(c) What is the column space of P?

(d) Find a positive definite matrix B such that $BP = P^t B$.

Problem 3. Let A be a real symmetric 2×2 matrix that is not a scalar multiple of the identify matrix. Consider the linear mapping L on the vector space of real 2×2 matrices defined by

$$L(X) = AX - XA.$$

Find the eigenvalues and corresponding eigenvectors of the linear mapping L.

Problem 4. (a) Let $\alpha = \text{Min}\left[\text{Max}_{|x| \le 1} \left|x^6 - x^3 - p_5(x)\right|\right]$, where the minimum is taken over all polynomials of degree ≤ 5. Find α and the polynomial $p_5(x)$ for which the minimum α is attained.

(b) Calculate the minimax approximation to the function $f(x) = |x|$ on $[-1, 1]$ by a quadratic polynomial.

Problem 5.

(a) For the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad a_n \ne 0$$

define

$$R = \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n|}$$

Show that every root x of $p(x) = 0$ satisfies

$$|x| \le \text{Max}\{R, \sqrt[n]{R}\}$$

(b) The Simpson's rule for computing the integral $I(f) = \int_a^b f(x)dx$ is

$$S(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad h = \frac{b-a}{2}.$$

Derive the error formula

$$I(f) - S(f) = -\frac{h^5}{90}f^{(4)}(\eta) \quad \eta \in [a, b]$$

for any $f \in C^4[a, b]$.

Problem 6. (a) Consider solving $y' = f(x, y)$ via the scheme

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(x_{n-1}, y_{n-1})$$

Determine its order and stability property.
(b) Find the region of absolute stability for the trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$