Doctoral Qualifying Exam: Applied Mathematics

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1. (a) Many state laws say that for each 10 m.p.h. (16 k.p.h) of speed you should stay at least one car length behind the car in front of you. Assuming that people obey this law (i.e., exactly one length), determine the density of cars as a function of speed (you may assume that average length of a car is 16 feet (5 meters). There is another law that gives a maximum speed limit (assume this is 50 m.p.h. (80 k.p.h.)). Find the flow (flux) of cars as a function of density.

   (b) The state laws above were developed in order to prescribe spacing between cars such that rear-end collisions could be avoided. Let α be the maximum deceleration rate (the same for all cars). Assume that the car ahead decelerates with the rate α, but the driver following takes τ seconds to react (after τ seconds this driver also decelerates with the rate α). How far back does a car have to be traveling at u m.p.h. in order to prevent a rear-end collision?

   (c) Discuss whether the law prescribed in part a) is appropriate for part b) if human reaction time is 1 second and the length of a car is as in part a).

2. *Polluting a river.* Consider a clean river flowing with velocity V in the $+x$ direction. Starting from time $t = 0$, pollutant is continuously dumped into the river at $x = 0$ in such a way that the concentration u of the pollutant at $x = 0$ is fixed and equal to P.

   (a) Assume that the pollutant does not diffuse. Write down an equation governing the concentration of pollutant for $x > 0$. Solve this equation and sketch the concentration of pollutant for a given $t > 0$.

   (b) Now, allow for the pollutant to diffuse with the diffusion constant D. Write down new equation that governs the concentration of the pollutant for all $t > 0$.

   (c) Rewrite this equation in the moving frame that travels with the velocity of the river. How is this equation called?

   (d) Sketch the solution for $t > 0$ if diffusion is dominant, and if diffusion is very weak. Find the conditions that D needs to satisfy for each of these two cases.

3. Consider the following homogeneous problem

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) - \lambda\frac{u}{x} = 0; \quad 0 < x < \infty$$

(a) Find the two independent solutions u_1 and u_2;

(b) Find the condition which λ needs to satisfy so that u_1 or u_2 are of finite s-norm in $[0, l)$ or (l, ∞). [For this problem, $s = 1/x$ and finite s-norm over D requires that $\int_D s|u|^2 dx < \infty$.]

(c) If the solutions of finite s-norm exist, construct the Green's function for this problem.

4. Let D be the region in the upper half plane ($y > 0$) exterior to the unit semi-circle. $D = \{(x, y)|\ y > 0,\ \ x^2 + y^2 > 1\}$.

(a) Use images to construct the Green's function satisfying:

$$\nabla^2 g = -\delta(x - \xi)\delta(y - \eta),$$

$$\frac{\partial g}{\partial n} = 0, \quad \text{on } \Gamma; \qquad g \sim a \ln r \quad \text{as } r \to \infty.$$

(b) Solve the BVP:

$$\nabla^2 u = 0,$$

$$\frac{\partial u}{\partial n} = \begin{cases} 0, & y = 0, \ \ |x| > 1 \\ f(\theta), & x^2 + y^2 = 1; \end{cases} \qquad u \ \text{ bounded as } \ r \to \infty.$$

5. (a) Consider the BVP for the biharmonic equation:

$$\nabla^4 u \equiv \nabla^2(\nabla^2 u) = 0, \quad \mathbf{x} \in V,$$

$$u = f(\mathbf{x}), \qquad \nabla^2 u = q(\mathbf{x}) \quad \text{on } S.$$

Express the solution in terms of the Green's function corresponding to the Dirichlet problem for Laplace's equation, i.e.

$$\nabla^2 g = -\delta(\mathbf{x}|\mathbf{x_0}), \quad \mathbf{x} \in V,$$

$$g = 0 \quad \text{on } S.$$

(b) Determine if the solution to the following BVP is unique:

$$\nabla^4 u = 0, \quad \mathbf{x} \in V,$$

$$u = f(\mathbf{x}), \qquad \frac{\partial}{\partial n}\left(\nabla^2 u\right) = 0 \quad \text{on } S.$$

6. Consider the BVP:

$$\nabla^2 u + k^2 u = 0, \qquad x \in (-\infty, \infty), \quad y \in (0, 1),$$

$$u(x, 0) = \begin{cases} 0, & |x| > 1 \\ f(x), & |x| \leq 1; \end{cases} \qquad u(x, 1) = 0.$$

The solution for u should also represent outgoing waves at $|x| \to \infty$. Assume that the constant k is real and $2\pi < k < 3\pi$.

Construct the Green's function for this problem, and use it to solve for u. Specifically identify all modes that represent outgoing waves as $|x| \to \infty$.