Doctoral Qualifying Exam: Linear Algebra and Numerical Methods

January 12, 2009

Problem 1. Let

$$M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad A = \frac{1}{25}M + 5M^T.$$

(a) Find $\mathbf{w} \in \mathbf{R}^2$ such that

$$||A\mathbf{w}|| = \max_{||\mathbf{x}||=1} ||A\mathbf{x}||$$

where $||\mathbf{v}||$ denotes the Euclidean length of $\mathbf{v}$. What is $||A\mathbf{w}||$?

(b) With $\mathbf{w}$ as found in part (a), find

$$\lim_{n\to\infty} ||A\mathbf{w}||^n \qquad \text{and} \qquad \lim_{n\to\infty} ||A^n\mathbf{w}||.$$

Problem 2. Let

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(a) Suppose that A is a 4×4 matrix having V as its null space. Find the reduced row echelon form of A.

(b) Find a matrix 4×4 matrix B for which V is both the null space and the left null space.

Problem 3. Suppose that $A \in \mathbf{R}^{n\times n}$ and that $A^TA = AA^T$.

(a) Prove for $\mathbf{x} \in \mathbf{R}^n$ that $||A\mathbf{x}|| = ||A^T\mathbf{x}||$.

(b) Show that if A is upper triangular then A is diagonal.

Problem 4. Let $x_0 < x_1 < \cdots < x_n$ be distinct real numbers, and y_i $i = 0, 1, \cdots, n$ be given real data values. Suppose that $s(x)$ is a piecewise polynomial which satisfies the following conditions:
(i) $s(x)$ is twice continuously differentiable on $[x_0, x_n]$;
(ii) $s(x)$ is a polynomial of degree ≤ 3 on each interval $[x_{i-1}, x_i]$, $i = 1, \cdots, n$; (iii) $s(x_i) = y_i$, $i = 0, 1, \cdots, n$, $s''(x_0) = 0$, $s''(x_n) = 0$.

(a) Show that

$$\int_{x_0}^{x_n} [s''(x)]^2 dx \le \int_{x_0}^{x_n} [g''(x)]^2 dx$$

where $g(x)$ is any twice continuously differentiable function on $[x_0, x_n]$ that satisfies the conditions $g(x_i) = y_i$, $i = 0, 1, \cdots, n$.
(b) Prove the uniqueness of $s(x)$.

Problem 5. (a) The iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ will converge to $\alpha = 1$ for some values of c [provided x_0 is chosen sufficiently close to α]. Find the values of c for which this is true. For what value of c will the convergence be quadratic?
(b) The trapezoidal rule for computing the integral $I(f) = \int_a^b f(x)dx$ is $T(f) = \frac{b-a}{2}[f(a) + f(b)]$. Derive the error formula $I(f) - T(f) = \frac{1}{2}\int_a^b f''(t)(t-a)(t-b)dt$ for any $f \in C^2[a, b]$.

Problem 6. Consider solving $y' = f(x, y)$ via the following explicit marching scheme

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h[b_0 y'_n + b_1 y'_{n-1} + b_2 y'_{n-2}]$$

where $y'_n = f(x_n, y_n)$.

(a) Show that if the scheme is fourth order then we must have $a_1 = 9$.
(b) Show that every such 4th-order scheme is unstable.