Doctoral Qualifying Exam : Real Analysis and Probability
Wednesday January 16, 2008

1. Consider a probability space $(\Omega, \mathcal{A}, P)$, and a sequence of events $\{A_n;\ \ n \geq 1\}$ therein.

(a) Prove the following version of the second Borel-Cantelli Lemma, which states : if the events A_n are *pairwise independent* and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then we must have $P(A_n \text{ i.o. }) = 1$.

(b) Suppose there exists events $B \in \mathcal{A}$ and $C \in \mathcal{A}$ such that $P(B) = P(C)$, and $B \subset A_n \subset C$ for all $n \geq N$ for some $N \in \{1, 2, 3, \cdots\}$. Prove that $P(A_n)$ converges as $n \to \infty$, and that

$$\lim_{n \to \infty} P(A_n) = P(B) = P(C).$$

2. Suppose $Z \in L^1$ is a random variable with $EZ < 0$, such that its moment generating function $E(e^{sZ}) := M_Z(s) < \infty$ for all $s \in (-a, b)$ for some $a > 0$ and $b > 0$. Prove that

$$P(Z \geq 0) \leq \rho := \inf_{0<s<b} M_Z(s),$$

and further that $\rho \in (0, 1)$.

3. **(a)** For random variables $\{X_n;\ n \geq 1\}$ and X defined on a common probability space, prove : $X_n \xrightarrow{\text{P}} X \Longrightarrow$ there exists a subsequence $n_k \to \infty$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$.

(b) Let Z be a random variable with a standard Normal distribution, and let $X_n := (-1)^n Z$ for $n \geq 1$. Examine whether X_n converges a) in distributionn ? b) in probability ? c) almost surely ?

4. Suppose that $s_n : [0, 2\pi] \to \mathbf{R}$ is a monotonically increasing step function for $n = 1, 2, \ldots$ and that the sequence $s_1(x), s_2(x), \ldots$ is monotonically increasing; that is,

$$x \geq y \quad \Rightarrow \quad s_n(x) \geq s_n(y)$$

and

$$n \geq m \quad \Rightarrow \quad s_n(x) \geq s_m(x).$$

Show that if s_n converges almost everywhere to a continuous function f on $[0, 2\pi]$ then s_n converges to f everywhere on $(0, 2\pi)$.

5. Suppose that $F \in L(\mathbf{R})$ and $f_n \in L(\mathbf{R})$ such that $|f_n| \leq F$ almost everywhere for $n \in \mathbf{N}$. Moreover, suppose that $|f_n(x)|$ converges to a function $f(x)$ almost everywhere. Define

$$\varphi_n(x) \;=\; \int_{-\infty}^{\infty} f_n(\xi) e^{-ix\xi}\, d\xi$$

and

$$\varphi(x) \;=\; \int_{-\infty}^{\infty} f(\xi) e^{-ix\xi}\, d\xi.$$

(a) Prove φ is continuous.

(b) Prove that φ_n converges to φ uniformly.

6. Determine which of the following functions are Lebesgue integrable over $\mathbf{R}$. Explain your determination.

Note: χ_S denotes the characteristic (or indicator) function of the set S.

$$(\mathbf{a})\quad f(x) = \frac{\sin^2 x}{x^2 - \pi^2} \qquad\qquad (\mathbf{b})\quad f(x) = \sum_{n=1}^{\infty} (-1)^n \chi_{[n, n+1/n^2]}(x)$$

$$(\mathbf{c})\quad f(x) = \frac{x^2}{1+x^2}\chi_{\mathbf{Q}}(x) \qquad\qquad (\mathbf{d})\quad f(x) = \frac{\sin^2 x}{1 + x^2 \sin^2 x}$$