Doctoral Qualifying Exam : Applied Mathematics
Wednesday January 16, 2008

1.) The van der Pol oscillator is an oscillator with nonlinear friction of small amplitude:

$$x'' + \epsilon x'(x^2 - 1) + x = 0,$$

in $t \geq 0$ with $\epsilon \ll 1$ and subject to the initial conditions $x = 1$ and $x' = 0$ at $t = 0$. The van der Pol oscillator is known to oscillate on a fast time ($\tau = t$) and drift in amplitude on a slow time ($T = \epsilon t$). Use the method of multiple scales to show that the amplitude of the oscillator eventually drifts to 2 for small, non-vanishing ϵ.

2.) Consider a reaction-diffusion system (first suggested by A. Turing in 1952, and later investigated by Prigogine and Lefever in 1968):

$$\frac{\partial x}{\partial \tau} = \kappa(1 - x) + \beta x(xy - 1) + \frac{\partial^2 x}{\partial s^2},$$

$$\frac{\partial y}{\partial \tau} = -\gamma x(xy - 1) + \frac{\partial^2 y}{\partial s^2},$$

where κ, β, and γ are positive constant coefficients, and s is the spatial variable. Find the spatially homogeneous steady state, and examine the linear stability of this steady state by first perturbing the homogeneous steady state with small normal-mode disturbances and writing down the linear equations for the small disturbances. What is the condition for instability when $O(\kappa) \ll O(\beta)$ and $O(\kappa) \ll O(\gamma)$.

3.) Consider the singular eigenvalue problem

$$\frac{d^2 u}{dx^2} + \lambda u = 0, \quad 0 < x < \infty$$

$$B_1 u = u'(0) = 0, \quad |u| < \infty \quad \text{as} \quad x \to \infty.$$

a) Find the eigenfunctions and eigenvalues for this problem.

b) Solve the related Green's function problem

$$\frac{d^2 G}{dx^2} + \lambda G = \delta(x - x'), \quad 0 < x < \infty$$

with the same boundary conditions as **a)**.

c) Integrate G over a large circle in the complex λ plane and determine the spectral representation of the delta function in terms of the "normalized" eigenfunctions of **a)**.

d) Use your result in part **c)** to deduce the Fourier cosine transform and its inverse. Is there a Parseval relation between the function and its transform?

4.) Consider the channel region defined by $D = \{(x,y) | \quad |x| < \infty, \quad 0 < y < 1\}$.

a) Find the Green's function satisfying

$$\nabla^2 G + 2ikM\frac{\partial G}{\partial x} + k^2(1-M^2)G = \delta(x-x')\delta(y-y'), \quad (x,y) \in D, \quad (x',y') \in D$$

$$\frac{\partial G}{\partial y} = 0, \quad y = 0, \quad G = 0, \quad y = 1, \qquad |x| < \infty$$

where $0 < M < 1$. This function should represent an outgoing wave as $|x| \to \infty$.

b) Solve the boundary value problem

$$\nabla^2 u + 2ikM\frac{\partial u}{\partial x} + k^2(1-M^2)u = 0, \quad (x,y) \in D$$

$$u_y(x,0) = f(x), \qquad u(x,1) = 0, \quad |x| < \infty,$$

where $f(x) = 0$ for $|x| > 1$.

c) Discuss the behavior of u as $x \to \pm\infty$.

5.) Consider the eigenvalue problem

$$\frac{d}{dx}(p(x)\frac{d}{dx}u) + [\lambda r(x) - q(x)]u = 0, \quad 0 < x < 1$$

$$\frac{d}{dx}u(0) = u(1) = 0, \quad x = 0, 1$$

where $p(x)$,$r(x)$, and $q(x)$ are positive and smooth on $0 < x < 1$.

Let $G(x,x')$ be the Green's function of the related problem

$$\frac{d}{dx}(p(x)\frac{d}{dx}G) - q(x)G = \delta(x-x'), \quad 0 < x, x' < 1$$

$$\frac{d}{dx}G(0,x') = G(1,x') = 0.$$

a) Show that:

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = -\int_0^1 r(x)G(x,x)\,dx.$$

b) Show also that:

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 \int_0^1 r(x)r(y)G(x,y)G(y,x)\,dx\,dy.$$

c) Use the result in **a)** for $p = r = 1$ and $q = 0$ to sum the series

$$\sum_{n=0}^{\infty} \frac{1}{\pi^2\,(2n+1)^2}.$$

6.) Solve the initial boundary value problem

$$\frac{\partial^2}{\partial t^2}u = \nabla^2 u, \quad |x| < \infty, \quad |y| < \infty, \quad z > 0, \quad t > 0$$

$$u = u_t = 0, \quad t = 0$$

$$u(x,y,0,t) = f(x,y)\,\sin\omega t, \quad t > 0,$$

where f has compact support in the $x - y$ plane and ω is a given positive constant, using the Green's function method.

Discuss the solution of this problem. In partucular, show that it becomes time harmonic for sufficiently large times.