Doctoral Qualifying Exam in Applied Mathematics
January 13, 2006

You have three hours for this exam, and each question carries equal weight. Show all work in the answer books provided. Good luck!

1) It is proposed to reintroduce a depleted stock of fish along a coastline by imposing a ban on fishing from the coast at $x = 0$ out to a distance $x = l$ at sea. A model for the scaled number density of fish u is

$$u_t = \nu u_{xx} + \Gamma u(1 - u) \quad x \in (0, l), \ t > 0$$

$$u_x(0, t) = 0, \quad u(l, t) = 0, \quad u(x, 0) = u_0(x).$$

Give a description or interpretation of this model for the fish population and state what the parameters ν and Γ represent.

Examine conditions for the growth or decay of a small population $u(x, t) \ll 1$ by linearizing the given problem for u with $u(x, 0) = u_0(x) \ll 1$. Write down the linearized problem and then solve it by putting $u(x, t) = e^{\Gamma t} v(x, t)$ and separating variables. Use your solution to show that in general the small population grows if

$$l > \frac{\pi}{2}\sqrt{\frac{\nu}{\Gamma}}.$$

What features of this result could, and what features could not, be established by dimensional analysis?

2) Consider the boundary value problem

$$u'' + u' = f(x) \quad x \in (0, 1)$$

$$B_1 u \equiv u'(0) + au(0) = c_1, \quad B_2 u \equiv u'(1) = c_2$$

where a is a real parameter. State whether the problem is self-adjoint, formally self-adjoint, or neither, and explain why.

Find the values of a for which a Green's function exists, and construct the Green's function when it exists. Give the solution of the boundary value problem in terms of the Green's function.

What condition must be satisfied by the data $(f(x), c_1, c_2)$ to ensure that a solution of the boundary value problem exists when $a = 0$. By considering the solution of the boundary value problem as $a \to 0$, can you identify the modified Green's function when $a = 0$?

3) In the boundary value problem

$$(L - \mu)u \equiv -u'' - \mu u = f(x) \quad x \in (0, 1)$$

$$u'(0) = c_1, \quad u'(1) = c_2$$

μ is a parameter. Write down the problem for the associated eigensystem of L and find the eigensystem explicitly. Use this to find the eigenfunction expansion of the solution to the boundary value problem for general μ. Identify the expression for the eigenfunction expansion of the Green's function.

What happens to the solution as μ approaches an eigenvalue, and under what conditions on the data $(f(x), c_1, c_2)$ does this *not* occur?

4) Solve the initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \ t > 0$$

$$\frac{\partial u}{\partial x} = -\sin t \quad x = 0, \ t > 0$$

$$u = 0 \quad x > 0, \ t = 0.$$

Show for $t \gg 1$ that the maximum of $u(0, t)$ does not coincide with the maximum of the forcing funtion $-\sin t$.

5) Find the Green's function G satisfying

$$\nabla^2 G = \delta(x - x')\delta(y - y')$$

in the upper half-plane $y > 0$ with the boundary condition

$$\frac{\partial G}{\partial y} - a\, G = 0 \quad \text{on } y = 0.$$

Here $a > 0$ is a constant and the source point (x', y') is in the upper half-plane, i.e., $y' > 0$. (Hint: Use a Fourier transform in x.)

Interpret your result in terms of images and show that when $a = 0$ it reduces to the case studied in class.

6) Solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + \delta(x)\delta(y)f(z)H(t)\sin\omega t$$

$$u = \frac{\partial u}{\partial t} = 0 \text{ when } t = 0 \text{ for } |\boldsymbol{x}| < \infty$$

where $f(z) = 1$ for $|z| < a/2$ and $f(z) = 0$ otherwise. Here $a > 0$ is a constant.

For fixed $\boldsymbol{x}$ show for sufficiently large time that

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{-a/2}^{a/2} \frac{\sin\omega(t - \rho)}{\rho} dz'$$

where $\rho = \sqrt{x^2 + y^2 + (z - z')^2}$.