**Ph.D. Qualifying Exam : Real Analysis and Probability - January 2005**

1. Suppose $\{X_1, X_2, \cdots\}$ is a sequence of independent random variables. Show that,

$$\sup_n X_n < \infty \text{ a.s. } \iff \sum_{n=1}^{\infty} P(X_n > A) < \infty, \text{ for some } A$$

2. **(i)** Show that

$$d(X,Y) := E\left(\frac{|X-Y|}{1+|X-Y|}\right)$$

defines a metric on the set of all random variables on the Borel real line, and that $d(X_n, X) \to 0$ as $n \to \infty$ if and only if $X_n \xrightarrow{P} X$.

**(ii)** Show that distance $d(X,Y)$ in (i) above in fact defines a metric space $(\mathcal{X}, d)$, where r.v.s are identified up to a.s. equivalence, by proving,

$$d(X,Y) = 0 \text{ if and only if } X = Y \text{ a.s.}$$

3. **(i)** State the (sub)martingale convergence theorem. Deduce the almost sure convergence of non-negative supermartingales, as a consequence.

**(ii)** An urn contains $r$ red and $g$ green balls. Balls are drawn randomly one at a time from the urn according to the following scheme. Each time we draw a ball; we note its color, replace it back and add $c$ more balls of the same color to the urn before the next draw. Let $X_n$ denote the fraction of green ball *after* $n$ draws. ($X_0$ denotes the fraction of greens before the first draw.)

   a) Prove that $\{X_n \,:\, n = 0, 1, 2, \cdots\}$ is a martingale.
   b) Find the probability that the $n$-th ball drawn is green.
   c) Prove that $X_n$ converges a.s. to some $X_\infty$. Can you say that $X_n$ converges in distribution to $X_\infty$ ? Justify your answer.

4. Assume that $f_n \to f$ uniformly on a set $S$. If each $f_n$ is continuous at a point $c$ in $S$, then prove that $f$ is also continuous at $c$.

5. Establish the validity of the following formulas:

(a) $x = 2\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n}$ if $-\pi < x < \pi$.

(b) $x^2 = \frac{\pi^2}{3} + 4\Sigma_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ if $-\pi \leq x \leq \pi$.

Be sure to justify convergence.

6. (a) Give an example of a sequence of functions $f_n$ which converge pointwise to 0, but do not converge in $L_p(\mathbf{R}, \mathbf{B}, \lambda)$ for any $p \geq 1$.

(b) Let $(X, \mathbf{X}, \mu)$ be a measure space. Suppose $f_n \to f$ almost everywhere where $f_n \in L_p$ and $f$ is measurable. Prove that if there exist $g \in L_p$ such that $|f_n| \leq g$ for all $n$ and $x$, then $f \in L_p$ and $f_n \to f$ in $L_p$.