Ph.D. Qualifying Exam in Analysis

January 12, 2005

1. Assume that $f_n \to f$ uniformly on a set S. If each f_n is continuous at a point c in S, then prove that f is also continuous at c.

2. Establish the validity of the following formulas:

   (a) $x = 2\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n}$ if $-\pi < x < \pi$.

   (b) $x^2 = \frac{\pi^2}{3} + 4\Sigma_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ if $-\pi \leq x \leq \pi$.

   Be sure to justify convergence.

3. (a) Give an example of a sequence of functions f_n which converge pointwise to 0, but do not converge in $L_p(\mathbf{R}, \mathbf{B}, \lambda)$ for any $p \geq 1$.

   (b) Let $(X, \mathbf{X}, \mu)$ be a measure space. Suppose $f_n \to f$ almost everywhere where $f_n \in L_p$ and f is measurable. Prove that if there exist $g \in L_p$ such that $|f_n| \leq g$ for all n and x, then $f \in L_p$ and $f_n \to f$ in L_p.

4. (a) Use contour integration to compute the integral

$$I = \int_0^\infty \frac{x}{1+x^3}\, dx.$$

   (b) **Outline** a method using contour integration to compute

$$I = \int_0^\infty \frac{x^m}{1+x^n}\, dx,$$

   for positive integers m, n with

$$n \geq m + 2 \tag{1}$$

   (i.e., generalize the result in (a)). Why is the condition (1) necessary?

5. (a) Use Rouche's Theorem to determine the number of roots of the equation $az^n = e^z$ inside $|z| = 1$. Consider only the cases (i) $a > e$ (ii) $a < e^{-1}$.

   (b) Use the argument principle to determine the number of zeros of $f(z) = z^4 + z^3 + 5z^2 + 2z + 4$ in the first quadrant.

6. Suppose f is entire and and $|f'(z)| \leq |z|$ for all z. Show that $f(z) = a + bz^2$ where a and b are constants with $|b| \leq 1/2$.