Ph.D. Qualifying Exam : Linear Algebra, Distribution Theory and Statistical Inference - January 2005

1. **(i)** Suppose random variables X and Y have a joint distribution such that $E(Y|X) = X$. Show that, we must then have,

$$\text{cov }(X, Y) = \text{ var } X, \text{ and } \text{ var } Y \geq 1.$$

(ii) Suppose $X \sim$ Uniform $(0, 1)$. Let a and b be arbitray fixed constants satisfying $0 < a < b < 1$. Consider the random variables,

$$Y := 1_{\{0<X<b\}}, \quad \text{and } Z := 1_{\{a<X<1\}}.$$

a) Y ans Z statistically independent ? Justify your answer.

b) Are there value(s) of a, b in $[0, 1]$, for which Y and Z are independent?

2. **(i)** Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n \geq 2$ from a Normal distribution $N(\mu, \sigma^2)$. Let,

$$\overline{X} = \frac{\sum_{i=1}^n X_i}{n}, \quad s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X})^2$$

be the sample mean and sample variance. Using moment generating functions, prove that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}.$$

(**Notation.** $\sim$ means "distributed as". χ^2_k denotes a chi-square variable, with k "degrees of freedom".)

(You may use : (i) the *additive property* of chi-squares, (ii) $Z \sim N(0,1) \Longrightarrow \chi^2_1$, and (iii) independence of the sample mean and sample variance, when observations are from a Normal parent distribution.)

(ii) Consider an exponential distribution (with a location parameter $\theta \in (-\infty, \infty)$, and scale parameter $\tau > 0$) having a density

$$f(x) = \begin{cases} \tau^{-1} \exp\{-(x-\theta)/\tau\}, & \text{if } x > \theta \\ 0, & \text{otherwise} \end{cases}$$

Based on a random sample $X_1, X_2, \cdots, X_n$ of size $n \geq 2$ from this distribution, find a *minimal sufficient statistic* for (θ, τ).

3. **(i)** Consider the exponential distribution in Problem 2(ii) above, with $\tau = 1$. Based on a random sample of size n, find the MLE and the UMVU estimators of the location parameter θ. Compare the two estimators for unbiasedness, variance and consistency.

   (ii) Consider a continuous distribution whose true density is $f(x)$; $x \in (-\infty, \infty)$. For the problem of testing

$$H_0 : \ f(x) = \pi^{-1/2} \exp(-x^2),$$

versus

$$H_1 : \ f(x) = \pi^{-1}(1 + x^2)^{-1};$$

   construct the *most powerful* (MP) size-α critical region, based on a single random sample X, and compute the corresponding *power*.

4. (a) Let D be the differential operator on polynomials $\phi(\omega) = \alpha_0 + \alpha_1\omega + \alpha_2\omega^2 + \alpha_3\omega^3 + \alpha_4\omega^4$ of degree ≤ 4. Thus $D\phi(\omega) = \alpha_1 + 2\alpha_2\omega + 3\alpha_3\omega^2 + 4\alpha_4\omega^3$. Show that D is a linear operator:

$$D\left(\alpha\phi(\omega) + \beta\psi(\omega)\right) = \alpha D\phi(\omega) + \beta D\psi(\omega)$$

   Given the isomorphism:

$$\alpha_0 + \alpha_1\omega + \ldots + \alpha_4\omega^4 \sim col(\alpha_0, \alpha_1, \ldots, \alpha_4)$$

   find a matrix representation of D.

   (b) Let L be the space of vectors made up of real-valued functions $\phi(t)$ at times $t = 0, \pm 1/3, \pm 2/3, \ldots$ satisfying the condition of periodicity $\phi(t+1) = \phi(t)$. What is the dimension of L?

5. Let

$$M = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 39 & 2 \\ 2 & 36 \end{bmatrix}.$$

   Solve the generalized eigenvalue problem $Kv^j = \lambda_j M v^j$ $(j = 1, 2)$, and find the matrix C such that $C^*MC = I$ and $C^*KC = diag(\lambda_1, \lambda_2)$. Given initial vectors $x(0)$ and $x'(0)$, show how the above results can be used to solve the initial-value problem:

$$Mx''(t) + Kx(t) = 0$$

6. (a) Write down the canonical Jordan matrix having eigenvalues $\lambda = 2, 2, 1, 3, 3$, one independent eigenvector for $\lambda = 2$, and two independent eigenvectors for $\lambda = 3$.

   (b) Consider the matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with eigenvectors $u^1, u^2, \ldots, u^n$. If $A = CDC^{-1}$ for some nonsingular matrix C, how are the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ and eigenvectors $v^1, v^2, \ldots, v^n$ of A related to those of D?