Ph.D. Qualifying Exam in Applied Mathematics

January 12, 2005

1. Two animal populations correspond to predator (P) and prey (ρ) populations that undergo periodic cycles in population levels. These populations are governed by the model equations

$$\begin{aligned} \frac{d\rho}{dt} &= \rho(a - bP), \\ \frac{dP}{dt} &= P(c\rho - d), \end{aligned}$$

where t is time and a, b, c, and d are positive constants.

   (a) Interpret the terms on the right-hand sides of these equations.

   (b) Specify the dimensions of the variables and of the parameters.

   (c) Nondimensionalize the variables and parameters in these equations and determine the dimensionless parameter(s). Interpret the meaning of these dimensionless parameter(s).

   (d) Determine the steady states of the model equations and determine their stability and classification.

   (e) Sketch the (ρ, P) phase plane, and include the nullclines, the steady states, the direction fields, and several typical trajectories.

   (f) What is the period of the small amplitude oscillations about the steady state, and what is the structural stability of the phase portrait of these model equations?

2. Consider the inhomogeneous boundary-value problem given by

$$\begin{aligned} x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y &= f(x), \quad 0 < x_0 < x_1, \\ u(x_0) = a, \qquad & u'(x_1) = b, \end{aligned}$$

for the Bessel function of order one-half with a forcing term $f(x)$ and inhomogeneous boundary conditions. For $x > 0$, two linearly independent solutions of the homogeneous equation are given by

$$y_1(x) = J_{1/2} = \left(\frac{2}{\pi x}\right)\text{sin}x, \quad y)_2(x) = J_{-1/2} = \left(\frac{2}{\pi x}\right)\text{cos}x.$$

Give a detailed explanation of how you would solve the inhomogeneous problem using this information (but do not carry out the mathematical steps).

3. Consider a boundary-value problem with a forcing term and homogeneous boundary conditions given by

$$\begin{aligned} u'' - u' &= f(x), \quad 0 < x < 1, \\ u(0) - u'(0) &= 0, \\ u(1) - u'(1) &= 0. \end{aligned}$$

   (a) If $f(x) = 0$, determine a nontrivial solution.

   (b) What consistency condition is required on $f(x)$ in order for a solution to exist?

   (c) Why does a modified Green's function have to be used for this problem? Explain.

   (d) Explicitly state the modified Green's function problem and explain the detailed steps needed to construct the modified Green's function for this problem (do not solve the mathematical problem).

   (e) Assuming the modified Green's function is given by $\mathcal{M}$, derive the general form of the solution u.

4. Heat flow in a one-dimensional rod is described by the initial-boundary-value problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \ t > 0, \\ u(x,0) &= 0, \\ u(0,t) &= 0, \quad u(1,t) + u_x(1,t) = 1, \ t > 0. \end{aligned}$$

   (a) Use separation of variables to determine the solution for this problem. Estimate the times t for which a truncation of this series solution is a good approximation.

   (b) Solve this problem using the Laplace transform in time. In particular, find the Laplace transform $U(s)$ of the variable $u(x,t)$ for this problem. Do not attempt to invert the Laplace transform using the Bromwich integral, but instead, expand the transform for large s, and then invert term by term where

$$\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{s}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-u^2}\,du.$$

Estimate the times t for which a truncation of this series solution is a good approximation.

5. A chemical with concentration $C(r,\theta)$ diffuses in a two-dimensional region in the shape of a half disk, which is defined by $x^2+y^2=r^2$ where $0 \le r \le 1$ and $0 \le y$.
   Use an eigenfunction expansion to determine the steady state distribution of the chemical concentration in this region if the concentration is $C=1$ along the circular edge, $r=1$, and $C=0$ along the bottom edge, $y=0$.

6. Use the method of images to determine the Green's function for Laplace's equation in each of the two regions below with homogeneous Dirichlet boundary conditions:

   (a) the half disk given in the previous problem.

   (b) the $45°$ wedge defined by $\pi/4 \le \theta \le \pi/2$. Note that the wedge goes off to infinity in the radial direction.