Ph.D. Qualifying Exam in Linear Algebra and Numerical Analysis

January 2005

1. (a) Let D be the differential operator on polynomials $\phi(\omega) = \alpha_0 + \alpha_1\omega + \alpha_2\omega^2 + \alpha_3\omega^3 + \alpha_4\omega^4$ of degree ≤ 4. Thus $D\phi(\omega) = \alpha_1 + 2\alpha_2\omega + 3\alpha_3\omega^2 + 4\alpha_4\omega^3$. Show that D is a linear operator:

$$D\left(\alpha\phi(\omega) + \beta\psi(\omega)\right) = \alpha D\phi(\omega) + \beta D\psi(\omega)$$

Given the isomorphism:

$$\alpha_0 + \alpha_1\omega + \ldots + \alpha_4\omega^4 \sim col(\alpha_0, \alpha_1, \ldots, \alpha_4)$$

find a matrix representation of D.

   (b) Let L be the space of vectors made up of real-valued functions $\phi(t)$ at times $t = 0, \pm 1/3, \pm 2/3, \ldots$ satisfying the condition of periodicity $\phi(t+1) = \phi(t)$. What is the dimension of L?

2. Let

$$M = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 39 & 2 \\ 2 & 36 \end{bmatrix}.$$

Solve the generalized eigenvalue problem $Kv^j = \lambda_j M v^j$ $(j = 1, 2)$, and find the matrix C such that $C^*MC = I$ and $C^*KC = diag(\lambda_1, \lambda_2)$. Given initial vectors $x(0)$ and $x'(0)$, show how the above results can be used to solve the initial-value problem:

$$Mx''(t) + Kx(t) = 0$$

3. (a) Write down the canonical Jordan matrix having eigenvalues $\lambda = 2, 2, 1, 3, 3$, one independent eigenvector for $\lambda = 2$, and two independent eigenvectors for $\lambda = 3$.

   (b) Consider the matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with eigenvectors $u^1, u^2, \ldots, u^n$. If $A = CDC^{-1}$ for some nonsingular matrix C, how are the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ and eigenvectors $v^1, v^2, \ldots, v^n$ of A related to those of D?

4. Consider the initial value problem

$$\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0. \end{aligned}$$

The following method has been proposed as a means of numerically approximating the solution to this equation:

$$y_{n+1} = -4y_n + 5y_{n-1} + 2hf(x_n, y_n) + 4hf(x_{n-1}, y_{n-1})$$

where h is the step size.

(a) What is the order of this method?

(b) Discuss the numerical stability of this method.

5. This problem pertains to the use of Newton's method and other closely related methods to find a root α of a continuous function f.

(a) Show that Newton's method has order of convergence 2 when α is a simple root of f.

(b) Recalling that Newton's method is derived from a linear Taylor polynomial approximation, derive the natural generalization of Newton's method that is based upon a quadratic Taylor polynomial approximation.

6. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that

$$M - N(h) = C_1 h + C_2 h^2 + C_3 h^3 + \cdots$$

where $C_1, C_2, C_3, \ldots$ are constants. Richardson extrapolation can be used to obtain improved approximations to M.

(a) Use the values $N(h)$ and $N\left(\frac{h}{3}\right)$ to produce an $\mathcal{O}(h^2)$ approximation to M.

(b) Use the values $N(h), N\left(\frac{h}{3}\right)$, and $N\left(\frac{h}{9}\right)$ to produce an $\mathcal{O}(h^3)$ approximation to M.