Some standard eigenvalue problems for $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$

1) Dirichlet boundary conditions $\phi(0) = 0,\ \phi(L) = 0$:

Eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$ normalized eigenfunctions $\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ with orthogonality relation $<\phi_m(x), \phi_n(x)>= \delta_{mn}$.

Series $f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$ where $b_n =< f(x), \phi_n(x) >$ and the inner product of $u(x)$ and $v(x)$ is $< u, v >= \int_0^L u(x)v(x)dx$.

2) Neumann or no-flux boundary conditions $\frac{d\phi}{dx}(0) = 0,\ \frac{d\phi}{dx}(L) = 0$:

Eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, \ldots$ normalized eigenfunctions $\phi_n(x) = \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}$ for $n = 1, 2, 3, \ldots$ but $\phi_0 = \sqrt{\frac{1}{L}}$ and $\lambda_0 = 0$ when $n = 0$. The orthogonality relation is $<\phi_m(x), \phi_n(x)>= \delta_{mn}$.

Series $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$ where $a_n =< f(x), \phi_n(x) >$ and the inner product of $u(x)$ and $v(x)$ is $< u, v >= \int_0^L u(x)v(x)dx$.

3) Periodic boundary conditions $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$:

Eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, \ldots$ the normalized eigenfunctions are

$$\phi_n^{(1)}(x) = \begin{cases} \frac{1}{\sqrt{L}} \cos \frac{n\pi x}{L} & \text{for } n = 1, 2, 3, \ldots \\ \frac{1}{\sqrt{2L}} & \text{when } n = 0 \ (\lambda_0 = 0) \end{cases}, \quad \phi_n^{(2)}(x) = \frac{1}{\sqrt{L}} \sin \frac{n\pi x}{L} \text{ for } n = 1, 2, 3, \ldots$$

with orthogonality relations

$$<\phi_m^{(1)}(x), \phi_n^{(2)}(x)>= 0,\ <\phi_m^{(1)}(x), \phi_n^{(1)}(x)>= \delta_{mn}, \text{ and } <\phi_m^{(2)}(x), \phi_n^{(2)}(x)>= \delta_{mn} .$$

Series $f(x) = \sum_{n=0}^{\infty} a_n \phi_n^{(1)}(x) + \sum_{n=1}^{\infty} b_n \phi_n^{(2)}(x)$ with coefficients $a_n =< f(x), \phi_n^{(1)}(x) >$, $b_n =< f(x), \phi_n^{(2)}(x) >$, and the inner product of $u(x)$ and $v(x)$ is $< u, v >= \int_{-L}^{L} u(x)v(x)dx$.