1. (a) Design an iteration $x_{n+1} = g(x_n)$ that converges to the reciprocal of some positive real number $a \in \mathbb{R}^+$, allowing computation of $1/a$ without using floating-point division. Hint: apply the Newton's method to the function $f(x) = 1/x - a$.

   (b) Determine *all* fixed points of this iteration, and classify their stability.

   (c) List at least two values of x_0 for which your iteration $x_{n+1} = g(x_n)$ does *not* converge to the fixed point $1/a$.

   (d) Show that the error of this method of approximating $1/a$ satisfies $|\epsilon_n| = a^n |\epsilon_0|^{2^n}$. Given $a = \pi$ and $x_0 = 1/2$, roughly how many steps are needed to approximate $1/a = 1/\pi$ with an absolute accuracy (tolerance) of 10^{-6}? (hint: you may use $\pi \approx 3$; estimate ϵ_1, ϵ_2, etc.).

2. Consider the following Runge-Kutta type method for approximating the solution to an autonomous initial value problem $\frac{dy}{dt} = f(y)$, $y_0 = y(0)$:

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{6}(F_1 + 4F_2 + F_3), \quad \text{where:} \\ F_1 &= f(y_n), \\ F_2 &= f\left(y_n + \frac{h}{2}F_1\right), \\ F_3 &= f\left(y_n - hF_1 + 2hF_2\right). \end{aligned}$$

   (a) For the test problem $f(y) = \lambda y$, find the numerical solution over a single time step of size h in the form $y_{n+1} = \phi(z) y_n$, where $z = \lambda h$. Analyze the function $\phi(z)$ to infer the order of accuracy of this method.

   (b) Write down the inequality defining the region of absolute stability of this method (you do *not* have to determine the interval of z ensuring stability). Is the method stable when solving $\frac{dy}{dt} = -2y$ with a step size of $h = 1$? What about $h = 1.5$?

3. Consider the following quadrature rule for approximating an integral of a sufficiently smooth function $f(x)$ over the interval $x \in [-1, 1]$:

$$I[f] \equiv \int_{-1}^{1} f(x)dx \;\approx\; Q[f] \equiv w_1 f\left(-\frac{2}{3}\right) + w_2 f(0) + w_1 f\left(\frac{2}{3}\right)$$

   (a) Find the weights w_1 and w_2 ensuring the highest possible accuracy of this rule, and express its error in terms of some derivative of $f(x)$ (use polynomial test functions to analyze the error). What is the method's degree of precision?

   (b) Compare the accuracy of this rule to the accuracy of the Simpson's rule, $S[f] \equiv \frac{1}{3}[f(-1) + 4f(0) + f(1)]$. You need to derive the error of the Simpson's rule to make this comparison (hint: both methods have the same degree of precision, but different error coefficients). Which method is more accurate?

   (c) Suppose you computed both $Q[f]$ and $S[f]$. Improve the accuracy of the quadrature using a linear combination of these two values (hint: examine $S[f] - Q[f]$).