Ph.D. Prelim: Exam. C
Probability Theory & Design of Experiments

May 22, 2024

Note that *a.e.* represents "almost everywhere"; $\stackrel{a.e.}{\to}$, $\stackrel{\mathbb{P}}{\to}$, $\stackrel{D}{\to}$, $\stackrel{v}{\to}$ represent convergence *a.e.*, in probability, in distribution, and vaguely respectively; $\mathbb{P}(A)$ represents the probability of event A; $\mathbb{E}X$ represents the expectation of a random variable X.

1. This section has two independent parts.

(a) (10 points) Prove that $1 + \mu([-2, 2]) \geq |\int_{-1}^{1} \varphi(t)dt|$, where μ is a probability measure on $(-\infty, \infty)$ and $\varphi(t)$ is the associated characteristic function. [HINT: You could work with the integral average of $\varphi(t)$ over $[-T, T]$, $T > 0$.]

(b) (10 points) Let F_n, G_n, F, G be distribution functions (df) and let $F_n \star G_n$ and $F \star G$ be the respective convolutions. Prove using characteristic functions that if $F_n \stackrel{v}{\to} F$ and $G_n \stackrel{v}{\to} G$, then $F_n \star G_n \stackrel{v}{\to} F \star G$. [Hint: You could work with independent random variable sequences $\{X_n\}, \{Y_n\}$ (with df F_n and G_n), and independent random variables X, Y (with df F and G).]

2. This section has two independent parts.

(a) (10 points) If $\{X_n, n \geq 1\}$ is a sequence of iid random variables and if $\mathbb{E}(X_1^+) = +\infty$ and $\mathbb{E}(X_1^-) < \infty$ then prove that $S_n/n \to +\infty$ *a.e.*, where $S_n = \sum_{j=1}^{n} X_j$.

(b) (10 points) Prove that even for a sequence $\{X_n, n \geq 1\}$ of independent random variables $X_n \stackrel{\mathbb{P}}{\to} 0$ does not imply that $S_n \stackrel{\mathbb{P}}{\to} 0$, where $S_n = \sum_{j=1}^{n} X_j$.

3. This section has two independent parts.

(a) (10 points) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that $\mathbb{P}(X_n = n^\alpha) = \mathbb{P}(X_n = -n^\alpha) = 1/2, (n = 1, 2, \ldots)$. State clearly under what conditions a CLT would hold and derive the resulting CLT.

(b) (10 points) Suppose that $\sup_n |X_n| \leq Y$ on Λ with $\int_\Lambda Y dP < \infty$. Here $\Lambda \in \mathcal{F}$. Show that

$$\int_\Lambda \left(\overline{\lim_{n \to 1}} X_n \right) dP \geq \overline{\lim_{n \to 1}} \int_\Lambda X_n dP.$$

4. (20 points) Consider a factorial experiment with two factors (A and B) with their interaction and n replicates, full model. Now suppose that to run this experiment a particular raw material is required. This raw material is available in batches that are not large enough to allow *all abn* treatment combinations to be run from the *same* batch. However, if a batch contains enough material for ab observations, then an alternative design is to run each of the n replicates using a separate batch of raw material. Consequently, the batches of raw material represent a randomization restriction or a **block**, and a single replicate of a complete factorial experiment is run within each block. (a) Write the model for this experiment, with appropriate constraints on the parameters. (b) Setup the normal equations and (c) solve them to give the estimators of the parameters and the residuals $e_{ijk}, i = 1, \ldots, a, j = 1, \ldots, b, k = 1, \ldots, n$.

5. (20 points) Consider the balance incomplete block design. Derive the interblock estimator $\tilde{\tau}_i$.

6. (20 points) Consider the single random factor model. Obtain the expected values of the $MS_{Treatment}$ and MS_{Error}, where MS represents Mean Square. Use these expected value results to obtain an unbiased estimator of σ_τ^2. What is the potential problem with the unbiased estimator of σ_τ^2 that you obtained?