Department of Mathematical Sciences
New Jersey Institute of Technology

Part A: Applied Mathematics — Doctoral Qualifying Exam, January 2018

The first three questions are based on Math 613 and the next three questions are about Math 651.

1. Consider the initial value problem

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 2 - x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(a) Sketch the characteristic diagram in x-t plane and indicate the region in which the solution is well-defined (i.e. does not break down).
(b) At what time and position does a shock form? Find the shock velocity.
(c) Sketch u versus x for several values of t before the breakdown of the solution.

2. Consider the small longitudinal vibration of a bar of density $\rho(x)$ and Young's modulus (stiffness) $E(x)$ governed by the linearized, small displacement $u(x,t)$, momentum equation

$$\rho(x)u_{tt} = (E(x)u_x)_x, \quad 0 < x < L, \quad t > 0$$

subject to boundary conditions $u(0) = u_x(L) = 0$.
(a) If a standing wave solution of the form $u(x,t) = w(x)e^{i\omega t}$ exists, where ω is called fundamental frequencies and $w(x)$ normal modes, then it must satisfy the governing equation and corresponding boundary conditions. Show that this results in a Sturm-Liouville eigenvalue problem.
(b) Estimate the smallest (non-zero) eigenvalue.

3. Consider a planar motion of a mass m that is attracted to the origin with a conservative force, F, inversely proportional to the square of the distance from the origin.
(a) Write down the expression for the difference between the kinetic energy ($T = mv^2/2$), where $v^2 = \dot{x}^2 + \dot{y}^2$, and the potential energy V, where $F = -\partial V/\partial r$, in polar coordinates (r, θ).
(b) Hamilton's principle asserts that the particle motion makes

$$J(r,\theta) = \int_{t_1}^{t_2} L(r, \dot{r}, \theta, \dot{\theta})dt$$

stationary, where L is the Lagrangian expressing the difference between the kinetic and potential energies. Use the Hamilton's principle above to derive the equations governing the trajectory of the particle. Do not attempt to solve the resulting equations.

4. (a) Consider the initial value problem

$$\dot{x} = |x|^{p/q},$$

where p and q are positive integers with no common factors, and $x(0) = 0$. First show that there is no periodic solution. Then show that there are an infinite number of solutions if $p < q$, and there is a unique solution if $p > q$.

(b) Find condition(s) on the constant coefficient α so there exists a Frobenius series solution about $x = 0$ for the differential equation

$$x^\alpha \frac{dy}{dx} = y.$$

5. Consider the linear differential operator

$$L(u) = -\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{\gamma u}{x^2} \text{ for } x \in (0, L),$$

with γ a constant coefficient.

(a) For the non-homoegeneous problem $L(u) = f(x)$ with boundary conditions $u(0) = 0$ and $u'(L) = 0$, find conditions on γ and $f(x)$ for the existence of a solution.

(b) For the eigenvalue problem $L(u) = \lambda u$ with boundary conditions $u(0) = 0$ and $u'(L) = \beta$ and λ the eigenvalue. Find conditions on γ and β so that $\lambda = 0$ is an eigenvalue, and find the corresponding eigenfunction. Further, prove that the eigenvalue λ is always positive when $\beta = 0$ and $\gamma \geq 0$.

6. Consider the heat equation with a source/sink (depending on the sign of α) on a circular ring formulated as

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} - \alpha u, \\
u(-L, t) &= u(L, t), \\
\frac{\partial u}{\partial x}(-L, t) &= \frac{\partial u}{\partial x}(L, t), \\
u(x, 0) &= g(x),
\end{aligned}$$

where the positive constant D is the diffusivity, $x = a\theta$, $L = a\pi$ with a the radius of the ring and $\theta \in (-\pi, \pi)$ the angle along the ring.

(a) Find the equilibrium solution, and discuss its dependence on the sign of α.

(b) For $\alpha > 0$, find the solution u in terms of the initial condition $g(x)$.

Useful Formulas

* Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}(p(x)\frac{d\phi}{dx}) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a < x < b$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0$$

* Rayleigh quotient:

$$\lambda = \frac{-p\phi d\phi/dx|_a^b + \int_a^b (p(d\phi/dx)^2 - q\phi^2)dx}{\int_a^b \sigma\phi^2 dx}$$

* Euler-Lagrange equation:

If $y_1, \cdots, y_n$ provide a local minimum for the functional

$$J(y_1, \cdots, y_n) = \int_a^b L(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

where $y_i \in C^2[a,b]$ and $y_i(a) = \alpha_i$ and $y_i(b) = \beta_i$, $i = 1, \cdots, n$, then y_i must satisfy the Euler-Lagrange system of n ordinary differential equations

$$L_{y_i} - \frac{dL_{y_i'}}{dx} = 0, \quad i = 1, \cdots, n, \quad x \in [a,b].$$